A Note on the Tight Simplification of Mechanisms*

Eduardo Perez-Richet
eduardo.perez@polytechnique.edu

Abstract

This note offers characterizations of tightness and weak tightness. It shows that when the preference domain is that of continuous utility functions on the outcome space, the two notions are equivalent to the outcome closure property of Milgrom (2010a).

1 Introduction

The theory of mechanism design often relies on direct mechanisms where the message space corresponds to the type space that describes the preferences of the agents and their private information. These spaces are often much richer than what can be reasonably implemented in practice. It is therefore important to know how to design simpler mechanisms without losing desirable theoretical properties. To address this question, Milgrom (2010a) introduces a notion of simplification that consists in restricting the messages available to the players. Simplification has the advantage that it can eliminate undesirable equilibria. However, it can also create new equilibria by eliminating profitable deviations. Milgrom (2010a) defines a simplification as *tight* if it does not create any new ε-Nash equilibrium. A weaker alternative is to rule out new Nash equilibria.

This note proposes characterizations of tightness. These characterizations are valid for any preference domain of the players over the outcome space. By specifying the preference domain, it is possible to produce characterizations of tightness that bear on the outcome function of the mechanism. For example, in Milgrom (2010a,b) the preference domain is the space of continuous utility functions. With this domain, I show that the outcome closure property of Milgrom (2010a) is not only sufficient but also necessary for tightness. I do this by proving that weak tightness (which is implied by tightness) implies the outcome closure property. This is the main result of this note. A byproduct of the demonstration is that there is no difference between weak tightness and tightness when the domain is that of continuous utility functions.

2 Setup

Let $N = \{1, \ldots, N\}$ be a set of players, and $\Omega \subseteq \Omega_1 \times \cdots \times \Omega_N$ the set of outcome profiles where Ω_n is the set of possible outcomes for player n. Together, they define an *environment*. A

*I'm grateful to Paul Milgrom for getting me interested in that topic.

mechanism $\mu = (X, \omega)$ for the environment (Ω, N) specifies a message space $X = X_1 \times \cdots \times X_N$ that defines the strategies available to the players, and an outcome function $\omega : X \to \Omega$. Taken together, a profile of utility functions over outcomes $u = (u_1, \ldots, u_N)$ where $u_n : \Omega_n \to \mathbb{R}$ and a mechanism $\mu(\Omega, N)$ characterize a game (μ, u). Denote by $\mathcal{U}$ a set of acceptable preference profiles over outcomes, and define a *complete environment* as a triple $\mathcal{E} = (\Omega, N, \mathcal{U})$.

Definition 1. *For a given environment (Ω, N), the mechanism $\mu' = (X', \omega')$ is a simplification of $\mu = (X, \omega)$ (and μ is an extension of μ') if for all $n \in N$, $X'_n \subseteq X_n$, and ω' is the restriction of ω to X': $\omega' = \omega_{|X'}$.*

A simplification is a mechanism that restrains the strategy space of the initial mechanism. By so doing it makes the expression of preferences less complicated, and it can eliminate undesirable equilibria. However, it can also create new Nash equilibria by eliminating profitable deviations. A tight simplification is one that does not have this undesirable feature.

Definition 2 (Weak Tightness). *The simplification μ' is weakly tight for $\mathcal{E} = (\Omega, N, \mathcal{U})$ if for every preference profile $u \in \mathcal{U}$, every pure strategy Nash equilibrium of (μ', u) is a pure strategy Nash equilibrium of (μ, u).*

For applications, it is sometimes useful to consider a stronger property.

Definition 3 (Tightness). *The simplification μ' is tight for $\mathcal{E} = (\Omega, N, \mathcal{U})$ if for every profile $u \in \mathcal{U}$ and every $\varepsilon \geq 0$, every pure strategy profile $x \in X$ that is an ε-Nash equilibrium of (μ', u) is also an ε-Nash equilibrium of (μ, u).*

Definition 4 (ε-Nash equilibrium). *For $\varepsilon \geq 0$, $x \in X$ is an ε-Nash equilibrium of a game (μ, u) if for each player n, and every strategy $x'_n \in X_n$,*

$$u_n\left(\omega_n(x'_n, x_{-n})\right) \leq u_n\left(\omega_n(x_n, x_{-n})\right) + \varepsilon.$$

A Nash equilibrium being a particular sort of ε-Nash equilibrium, it is clear that tightness implies weak tightness.

3 Characterizations

A simplification satisfies the *deviation conservation property* if for every strategy profile in the restricted set that is not a Nash equilibrium of the extended game, there exists one player with a profitable deviation in her restricted strategy set. Denoting by $NE(\Gamma)$ the set of pure strategy Nash equilibria of a complete information game Γ, the definition can be formally stated as follows.

Definition 5 (**DCP**). *Given $\mathcal{E} = (\Omega, N, \mathcal{U})$, the mechanism μ' satisfies **(DCP)** with respect to μ if*

$$\big(\forall u \in \mathcal{U}\big)\big(\forall x' \in X' \smallsetminus NE(\mu, \succ)\big)\big(\exists n \in N\big)\big(\exists \tilde{x}_n \in X'_n\big)\; u_n(\omega_n(\tilde{x}_n, x'_{-n})) > u_n(\omega_n(x')).$$

This is a straightforward characterization of tightness: a pure strategy Nash equilibrium of a game is a strategy profile that no player wants to deviate from. It uses game theoretic and equilibrium concepts, and it would be more attractive to have a characterization that uses only preference related concepts. In order to do that, and for a given preference profile $u \in \mathcal{U}$, define the *upper-contour* of a strategy profile x for player n as

$$U_n(x) = \left\{ (\tilde{x}_n, x_{-n}) \middle| \Big(\tilde{x}_n \in X_n\Big) \text{ and } \Big(u_n(\omega_n(\tilde{x}_n, x_{-n})) > u_n(\omega_n(x))\Big) \right\} \subseteq X.$$

It is the set of strategy profiles of the extended game that are strictly preferred to x by player n and that she can reach by a unilateral deviation from x. Define the *upper-contour set* of x as

$$U(x) = \bigcup_{n \in N} U_n(x).$$

It is the set of strategy profiles in X that are preferred to x by some player and can be reached unilaterally by this player. I can then define the *Upper-Contour Closure Property.*

Definition 6 (**UCCP**). *Given $\mathcal{E} = (X, N, \mathcal{U})$, the mechanism μ' satisfies **(UCCP)** with respect to μ if*

$$(\forall u \in \mathcal{U})(\forall x' \in X') \ \Big(U(x') = \emptyset \text{ or } U(x') \cap X' \neq \emptyset\Big).$$

Theorem 1. *The following statements are equivalent:*

(i) μ' is weakly tight for $\mathcal{E}$.

*(ii) μ' satisfies **(DCP)**.*

*(iii) μ' satisfies **(UCCP)**.*

Proof. $(i) \Leftrightarrow (ii)$. **(DCP)** says that for any strategy profile of the restricted game from which a player would have a profitable deviation in the extended game, there is a player with a profitable deviation in the restricted game. Therefore, it is clear that no new pure strategy Nash equilibrium can be created by the restriction, proving the sufficiency of the property. Necessity is also true for if **(DCP)** did not hold, there would be a restricted strategy profile that is not a pure strategy Nash equilibrium of the extended game and from which no player would be willing to deviate in the restricted game, ie. a new pure strategy Nash equilibrium of the restricted game.

$(ii) \Leftrightarrow (iii)$. Suppose that μ' satisfies **(DCP)** and let $x' \in X'$. If x' is a pure strategy Nash equilibrium of the extended game, then for every player n, $U_n(x') = \emptyset$ and $U(x') = \emptyset$. If x' is not a pure strategy Nash equilibrium of the extended game, by **(DCP)** there exists a player n with a deviation $x_n \in X'_n$ from x'. But then $(x_n, x'_{-n)} \in \Big(U_n(x') \cap X'\Big) \subseteq \Big(U(x') \cap X'\Big)$. This shows necessity.

Suppose now that μ' satisfies **(UCCP)** and consider a strategy profile $x' \in X'$ that is not a pure strategy Nash equilibrium of the extended game. Then $U(x') \neq \emptyset$. By **(UCCP)** it is possible to pick a strategy profile $x \in U(x') \cap X'$, implying that there is some n such that $x \in U_n(x') \cap X'$. Then $x = (x_n, x'_{-n})$ where $x_n \in X'_n$ is a profitable deviation from x'_n for player n, and this concludes. □

It is easy to offer a similar characterization of tightness and I do not write it down to save space.

4 A sufficient but non-necessary condition

An earlier version of Milgrom (2010a) defined the *best-reply closure property*, and proved it to be sufficient for weak tightness. A simplification satisfies this property if, to any strategy profile of her competitors that lies in the restricted set, a player can best-respond with a strategy in her restricted set.

Definition 7 (**BRCP**)**.** *Given $\mathcal{E} = (\Omega, N, \mathcal{U})$, the mechanism μ' has the best-reply closure property with respect to μ if*

$$(\forall u \in \mathcal{U})(\forall n \in N)(\forall x'_{-n} \in X'_{-n}) \left(\arg\max_{x_n \in X_n} u_n(\omega_n(x_n, x'_{-n})) \right) \cap X'_n \neq \emptyset.$$

One drawback of **(BRCP)** is that best replies do not always exist. The definition is always correct because when a maximum does not exist the maximizing set is empty. But it is possible for a strategy profile not to be a Nash equilibrium even though no player has a best reply to this profile, and this justifies the use of deviations rather than best replies. However, this is not the reason why **(BRCP)** is only sufficient. Indeed, the counter example that follows uses finite strategy sets for which best replies always exist. In fact, **(BRCP)** is too strong because it assumes that every player conserves a best reply. First I reproduce the simplification theorem since it has disappeared from the final version of Milgrom (2010a).

Theorem 2 (Milgrom)**.** *If μ' satisfies **(BRCP)**, then it is weakly tight.*

Proof. Suppose **(BRCP)** is satisfied. Suppose that $x' \in X'$ is not a Nash equilibrium of the extended game. **(BRCP)** implies that every player has a best-reply to x' that lies in X', and the fact that x' is not an equilibrium of the extended game implies that for at least one player, this best-reply constitutes a strict improvement over x'. Hence **(DCP)** is satisfied. □

Example 1. Consider the following game.

	L	C	R
U	$(0,2)$	$(2,0)$	$(1,1)$
D	$(1,-1)$	$(1,0)$	$(2,1)$

Its unique pure strategy equilibrium is (D, R). Consider the simplification that eliminates L for player 2. It does not satisfy **(BRCP)** since the best-reply of 2 to U in the extended game is L, which is no longer available in the restricted game. However, no new pure strategy equilibrium is created by the simplification.

5 Tightness and the Outcome Closure Property

Milgrom (2010a) uses restrictions on $\mathcal{U}$ to define conditions on the outcome function that ensure tightness. For this purpose, I endow each Ω_n with a topology $\mathcal{T}_n$ and define $\mathcal{C}_n$ to be the set of continuous functions from Ω_n to $\mathbb{R}$. Let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$. The *Outcome Closure Property* is defined as follows.

Definition 8 (OCP). *A simplification $\mu' = (X', \omega')$ of the mechanism $\mu = (X, \omega)$ satisfies **(OCP)** if for every player n, every profile $x'_{-n} \in X'_{-n}$, every $x_n \in X_n$, and every open neighborhood $\mathcal{O}$ of $\omega_n(x'_{-n}, x_n)$, there exists $x'_n \in X'_n$ such that $\omega_n(x') \in \mathcal{O}$.*

(OCP) means that if a given outcome is reachable by a player when other players play according to strategies in the simplified set, then she can approach it as closely as desired by picking strategies in her restricted set. Using the language of topology, it says that for every n, the space $\omega_n(X')$ is dense in the space $\omega_n(X_n, X'_{-n})$. Milgrom (2010a) proves that if μ' satisfies **(OCP)**, then it is tight. I show a more general result under the slight restriction that each $(\Omega_n, \mathcal{T}_n)$ is metrizable[1] with a distance d_n.

Theorem 3. *If each $(\Omega_n, \mathcal{T}_n)$ is metrizable with a distance d_n, the following statements are equivalent*

*(i) μ' satisfies **(OCP)**.*

(ii) μ' is a tight simplification of μ.

(iii) μ' is a weakly tight simplification of μ.

Proof. Milgrom (2010a) proves that (i) implies (ii), and it is obvious that (ii) implies (iii). Therefore I need only show that (iii) implies (i). To do that, I show that if μ' does not satisfy **(OCP)**, then it is not weakly tight. Suppose indeed that (i) is not true. Then there exists some n, some profile $x'_{-n} \in X'_{-n}$, and some strategy $x_n \in X_n$ such that for a certain neighborhood $\mathcal{O}$ of $\tilde{\omega}_n = \omega_n(x_n, x'_{-n})$, and every $x'_n \in X'_n$, $\omega_n(x'_n, x'_{-n}) \notin \mathcal{O}$. Since $\mathcal{O}$ is an open neighborhood, there must exist some $r > 0$ such that the ball $\mathcal{B}(\tilde{\omega}_n, r) \subsetneq \mathcal{O}$. Then for every $\hat{\omega}_n \in \Omega_n$, let $u_n(\hat{\omega}_n) = r - d_n(\tilde{\omega}, \hat{\omega}_n)$ if $\hat{\omega}_n \in \mathcal{B}(\tilde{\omega}_n, r)$, and otherwise $u_n(\hat{\omega}_n) = 0$. u_n is continuous by construction since the distance function is continuous. Let the other utility functions be uniformly equal to 0, so that they are continuous as well. In the simplified game associated with this utility profile, player n cannot get a utility higher than 0 when the other players use the profile x'_{-n}. Therefore, for any $x'_n \in X'_n$, the profile (x'_n, x'_{-n}) is an equilibrium of the simplified game. However, it is clear that none of these strategy profiles is a Nash equilibrium of the initial game as player n would be better off by playing $x_n \notin X'_n$, which violates weak tightness. □

References

Milgrom P., 2010a. Simplified Mechanisms with Applications to Sponsored-Search Auctions. Games and Economic Behavior, 70:1, 62-70.

Milgrom, P., 2010b. Assignment Messages and Exchanges. American Economic Journal: Microeconomics, 1:2, 915-113.

[1] A metrizable space is a topological space that is homeomorphic to a metric space. This includes countable spaces and vector spaces with their usual topology, as well as their products, covering most applications.