Test Design under Falsification[*]

Eduardo Perez-Richet[†] Vasiliki Skreta [‡]

December 5, 2017

Abstract

We derive an optimal test when cheating is possible in the form of type falsification. Optimal design exploits the following trade-off: while cheating may lead to better grades, it devalues their meaning. We show that optimal tests can be derived among cheating-proof ones. Our optimal test has a single 'failing' grade, and a continuum of 'passing' grades. It makes the agent indifferent across all moderate levels of cheating. Good types never fail, but bad types may pass. An optimal test delivers at least half of the full information value. A three-grade optimal test also performs well.

Keywords: Information Design, Falsification, Tests, Cheating, Persuasion.
JEL classification: C72; D82.

[*] We thank Ricardo Alonso, Philippe Jehiel, Ines Moreno de Barreda, Meg Meyer, Philip Strack and Peter Sorensen for helpful comments and suggestions. Eduardo Perez-Richet acknowledges funding by the ANR (ANR STRATCOM - 16-TERC-0010-01). Vasiliki Skreta acknowledges funding by the European Research Council (ERC) consolidator grant "Frontiers In Design."

[†] Sciences Po, CEPR – e-mail: eduardo.perez@sciencespo.fr

[‡] UT Austin, UCL, CEPR – e-mail: vskreta@gmail.com

1 Introduction

Tests are prevalent, and stakes are often high for all concerned parties. Teachers prepare their students to pass tests in order to gain admission to selective schools and universities. Issuers seek to obtain a good rating for their assets. Pharmaceutical companies seek FDA's approval for new drugs. Car manufacturers need to have their vehicles pass emission tests. The list is suggestive of how wide-ranging and relevant tests are, and why it is important that test results are reliable: Fairness, inadequacy, financial distraught, and environmental pollution are at stake when tests are compromised.

However, cheating is equally prevalent, and often successful. It is common in standardised graduate admission tests. Pharmaceuticals have come under scrutiny for using sub-standard clinical trial designs in order to obtain FDA's approval as in Sarepta's case (*The Economist*, October 15, 2016).[1] Car manufacturers sometimes cheat on pollution emission tests and have been subjected to substantial fines as a result.

Yet, there has been no study of how to design tests optimally in the presence of cheating. This is the first paper to do so. We model the situation as a three-player interaction between a principal, an agent, and a decision maker. The agent—a professor, a school, an asset issuer, a car manufacturer, or the car industry—is endowed with multiple items—students, assets, or car models—to be tested in order to gain approval by the decision maker. The agent would like all items to be approved unconditionally, whereas the decision maker–or several identical decision makers, employers, investors, consumers–wish to approve items selectively, depending on their hidden type. To uncover the types of the items, the principal, whose interests are aligned with those of the decision maker(s), designs a test to which each item is subjected. This test is modelled as a Blackwell experiment: a probability distribution over signals (test results, grades) as a function of the type of an item. The decision maker decides after observing these signals, and thus does not commit in advance to an approval policy contingent on signals.

The agent has a cheating technology at his disposal. He can, possibly at a cost, falsify the type of some of his items for testing purposes, so that, for example, 'bad' items generate the same signal distribution as 'good' items. By doing so, he garbles the information generated by the test for the decision maker. The decision maker can learn about the cheating strategy of

[1] http://www.economist.com/news/leaders/21708726-approving-unproven-drug-sets-worrying-precedent-bad-

the agent from the realized cross-sectional distribution of test results.[2] As a consequence, the decision maker can respond to on and off-equilibrium path cheating by altering the beliefs she associates to different test results. Lack of commitment implies that these changes in beliefs are the only tool to discipline cheating by the agent.

The way Volkswagen compromised emission tests[3] is a good illustration of our cheating technology, as the following quote reveals. On January 11, 2017, "VW agreed to pay a criminal fine of $4.3bn for selling around 500,000 cars fitted with so-called "defeat devices" that are designed to reduce emissions of nitrogen oxide (NOx) under test conditions." Just a day after that, the US Environmental Protection Agency (EPA) accused Fiat Chrysler Automobile of using illegal software in conjunction with the engines which, allowed thousand of vehicles to exceed legal limits of toxic emissions.[4], [5], [6] Another example would be schools deciding to teach their students to the test, thus making bad students appear good.

The model, while stylized, captures a key trade-off: cheating can increase the rate of approval, by increasing the chance that "bad" items generate good test results, but too much of it can make test results so unreliable that it nullifies approvals. So, even if cheating bears no cost, or punishment, excessive cheating can hurt the agent, and a rational cheater, therefore, manipulates by not cheating too much.[7] Cheating complicates test design, as one has to take into account how the agent's cheating strategies counteract the principal's information design. Our analysis shows how the principal can exploit the aforementioned trade-off to design informative tests in spite of cheating, and even in the absence of explicit punishments or unrealistic commitment on the side of the decision maker.

In our model, the agent has a continuum of items, each of which is independently either good or bad, with the same probability. The decision maker wishes to approve good items, and reject bad ones. The prior probability that an item is of the good type, μ_0, is below the

[2] More precisely, we assume a continuum of items with independently and identically distributed types, so, by the law of large numbers, this cross-sectional distribution is deterministic and it partially reveals the falsification strategy of the agent.

[3] https://en.wikipedia.org/wiki/Volkswagen_emissions_scandal

[4] http://www.economist.com/news/briefing/21667918-systematic-fraud-worlds-biggest-carmaker-threatens-

[5] http://www.economist.com/news/business-and-finance/21714583-after-volkswagen-agrees-large-criminal-

[6] http://www.economist.com/blogs/graphicdetail/2017/01/daily-chart-13

[7] Cheaters on standardized tests for graduate admissions (GRE's) are aware of this trade-off, and advise each other in online forums to make a strategic number of mistakes: *'. . . "We must follow the score-control strategy," admonishes one. Test-takers were advised to make five mistakes to ensure scores aren't so high that they expose the system. . . .'* See http://www.reuters.com/article/us-china-testing-cheating-idUSTRE76Q19R20110727.

decision maker's approval threshold $\hat{\mu}$. A cheating strategy is a choice of falsification rates p_B, the share of bad items to be masqueraded as good ones, and p_G, the share of good items to be disguised as bad ones. While this cheating technology allows the agent to garble the information generated by the test, and to turn any test completely uninformative, it does not make all garbles available.[8] This limitation of available garbles helps only if the set of signals generated by the test is sufficiently rich. Indeed, the agent can garble any sufficiently informative binary test (such as the fully informative one) into his optimal information structure, hence optimal tests must use more than two signals.

The optimal test we derive has a number of remarkable features and delivers some practical insights. First, it is cheating-proof in the sense that it does not give the agent any incentive to cheat. Second, despite the fact that there are only two actions to take, it is "rich" in the sense that it generates a continuum of signals, only one of which leads to rejection, while a continuum of signals are associated with approval. Hence, the *receiver side* revelation principle that usually holds in Bayesian persuasion (Kamenica and Gentzkow, 2011) and mediation problems (Myerson, 1991, Chapter 6), which allows to reduce the information design problem to the problem of designing a recommendation system, does not hold in our environment. Third, all items that would be approved under full information are approved under our optimal test, but some items that should be rejected are also approved. That is, our optimal test leads to some type II errors, but no type I errors. Fourth, it is ex-ante Pareto efficient, and gives the decision maker at least 50% of the payoff she would get under full information. Fifth, the distribution of signals generated by the good type first-order stochastically dominates that generated by the bad type. Furthermore, our optimal test makes the agent indifferent between no cheating, and any other approval threshold he could induce through cheating.

To see why tests with more signals can be beneficial, it is useful to consider adding a third "noisy" signal to the fully informative test. One can choose the probabilities that the good and bad type generate this signal so that, in the absence of cheating, it leads the decision maker to a belief equal to the approval threshold $\hat{\mu}$. With such a test, any amount of falsification leads the decision maker to lower the belief associated with the intermediate signal, and thus reject items that generate this signal. Then the agent has to weigh the benefit of cheating (bad types are

[8] If all garbles were attainable, the agent could garble any sufficiently informative test into his optimal information structure—the one he would pick if he were the information designer, thus making the principal useless.

more likely to generate the top signal), with its endogenous cost (losing the mass of items that generate the intermediate signals). To make such a test as good as possible for the decision maker, the principal can choose the test so that these two effects compensate each other, thus making the agent indifferent between his optimal amount of falsification, and no falsification. The resulting test is cheating-proof, and generates valuable information for the decision maker.

In fact, we establish a general *no-falsification principle*, which shows that, for any test, there is an equivalent cheating-proof test that generates the same information and payoffs to all parties. This result echoes the revelation principle but has some additional subtleties. Combined with the representation of experiments as convex functions introduced in Kolotilin (2016), and further studied in Gentzkow and Kamenica (2016b), it allows us to reformulate the optimal design problem of the principal as a maximization problem over convex functions representing tests, under a no-cheating incentive constraint. The no-cheating incentive constraint can be formulated as a condition bearing on the payoff of approval thresholds induced by cheating. We show that there exists a unique test such that, first, there is a single reject signal generated by the bad type only, and, second, the agent is indifferent between not cheating, and inducing any other approval threshold through cheating. This test is characterized by a differential equation that we can solve in closed form. We then show that this test is in fact optimal.

When falsification is costly, the no-falsification principle holds if the marginal cost of increasing p_B does not increase too fast. We show that the fully informative test is optimal whenever the cost is sufficiently high. When it is not, we derive the optimal test under a linear cost function, and show that it satisfies the same properties as without cost. Furthermore, our optimal test becomes more informative as cheating becomes more costly. In Appendix C, we show how to find an optimal test for a larger class of cost functions.

We first derive optimal tests under two auxiliary conditions that we later relax: The first one is that (possibly costly) falsification is perfectly observable, and the second is that falsification rates are constrained so that $p_B + p_G \leq 1$. The latter constraint rules out falsification rates so high that they would lead to an inversion of the meaning of signals. Both assumptions are useful in allowing us to focus on the main trade-offs, and can be compelling in some cases but not always, so we relax them in Section 9.

When perfect observability is relaxed, the decision maker can still partially infer cheating behavior from the cross-sectional distribution of signals. We show that, as long as falsification is costly, among all falsification rates that generate the same information set for the decision maker, one strictly dominates all the other. Therefore, in a subgame perfect equilibrium, conditional on reaching a certain information set, the decision maker knows for sure what choice the agent must have made, and can adopt the same beliefs as in the case of perfect observability. This is true for information sets both on and off the equilibrium path. Therefore, all results in the costly case still hold when the auxiliary assumptions are relaxed. For the costless case, they extend through two arguments. The first one is a selection argument. By taking a falsification cost that converges to 0, we obtain our optimal test in the costless case. The second argument relies on the idea that the agent, conditional on attaining any given payoff, should prefer lower falsification rates. This can be nicely captured by assuming that the agent has lexicographic preferences, with approval rate as its first dimension, and any decreasing function of p_B, and p_G on the second dimension. Under such lexicographic preferences, the dominance argument holds as well, implying that our optimal test in the costless case is optimal in this relaxed setup as well.

2 Related Literature

Theoretical work on Bayesian Persuasion. We introduce cheating in the information design literature. Kamenica and Gentzkow (2011) examine a party (sender) who wishes to design the best way to disclose information so as to persuade a decision-maker who may have different objectives.[9] In our approach, the information designer acts in the interest of the receiver, but the persuader may tamper with the chosen experiment by falsifying the state.

This paper is closely related to recent works that study Bayesian persuasion in the presence of moral hazard. In Boleslavsky and Kim (2017), Rodina (2016), and Rodina and Farragut (2016), the prior distribution of the state is endogenous and depends of the agent's effort. The aforementioned papers differ in the principal's objective. Related to these works is Hörner and Lambert (2016), who find the rating system that maximizes the agent's effort in a dynamic model where the agent seeks to be promoted. In Rosar (2017) the principal designs a test that the agent decides whether or not to take. In our paper, participation to the test is not optional, and the agent cannot alter the distribution of types, but he can tamper with the test itself.

[9] There are several extensions of this leading paradigm including Gentzkow and Kamenica (2014), who allow for costly signals and Gentzkow and Kamenica (2016a) where two senders "compete" to persuade.

We also relate to Bizzotto, Rudiger, and Vigier (2016) and to Cohn, Rajan, and Strobl (2016), since there, like in our paper, certifiers designing tests need to take into account the fact that firms are not passive, but react to the certification environment. In Bizzotto et al. (2016) agents choose what additional information to disclose, whereas we investigate what happens when firms cheat.

Our analysis is somewhat reminiscent to that of recent papers that study optimal information design in specific contexts. Chassang and Ortner (2016) design the optimal wage scheme to eliminate collusion between an agent and the monitor. The optimal wage scheme is similar to the buyer-optimal signal in Condorelli and Szentes (2016). In that paper as well as in Roesler and Szentes (2017), the buyer optimal signal is such that the seller is indifferent across all prices he can set. Our paper uncovers a similar property, as the optimal test makes the agent indifferent across all moderate falsification levels.

On the technical side, we represent experiments as convex functions as in Kolotilin (2016) and Gentzkow and Kamenica (2016b). The latter study costly persuasion in a setup where the decision-maker cares only about the expectation of the state of the world. In our setup the principal's decision also depends on a single-dimensional object: his belief that the state is good.

Costly state falsification/Hidden income/Hidden Trades. Lacker and Weinberg (1989) incorporate costly state falsification in a risk-sharing model. Cunningham and Moreno de Barreda (2015) model cheating as costly state falsification in a context similar to ours, but they study equilibrium properties under a fixed testing technology, whereas we focus on optimal test design. Hidden trades can also be viewed as a form of cheating and are studied in Golosov and Tsyvinski (2007), and references therein. Grochulski (2007) models tax avoidance using a general income concealment technology analogous to the costly state falsification technology of Lacker and Weinberg (1989). In Landier and Plantin (2016), agents can hide part of their income which can be interpreted both as tax evasion and as tax avoidance.

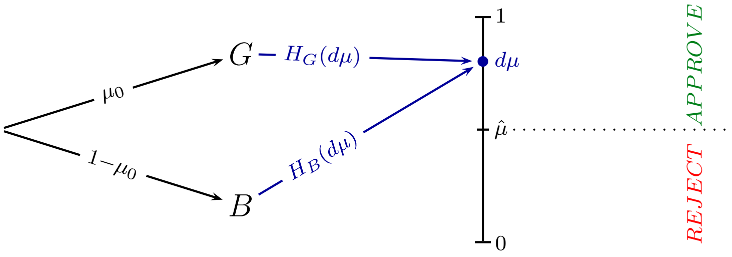


Figure 1: *A test is modelled as a Blackwell experiment. We normalize tests by equating signals to beliefs.*

3 Model

There are three players: a principal (she), who designs a test, an agent (he) endowed with a continuum of ex ante identical items to be tested, and a decision maker (also she), who decides whether to approve or reject each of the items. Items are indexed by $i \in [0, 1]$ and can be either good or bad, $t_i \in \{G, B\}$. The common prior is that all items are identically and independently distributed with probability μ_0 that any given item is good.

The agent wants each of his items to be approved. We normalize his payoff from an approval to 1, and that from a rejection to 0. The principal and the decision maker have identical preferences. They would like to approve only good items. Their payoff is $g > 0$ for approving a good item, and $-b < 0$ for approving a bad one. Without loss of generality, their rejection payoff is normalized to 0. Then, the decision maker approves an item if she believes that it is good with probability greater than (or equal to) the threshold $\hat{\mu} = \frac{b}{g+b}$. We assume that she approves an item whenever she is indifferent.[10]

Tests. To learn about the items, the principal designs a test that each item is subjected to. We describe a test as a Blackwell experiment (Blackwell, 1951, 1953): a measurable space of signals Σ, and probability measures H_G and H_B on Σ. Signal realization σ^i induces a belief μ^i through Bayes' rule, where $\mu^i \in [0, 1]$ is the updated probability that i is good. The approval decision of the decision maker for each item and, hence, the final payoffs of the three players

[10] Our analysis can be easily adapted to the case of an agent with distinct approval values for good and bad items.

only depend on the belief μ^i that the test induces for each item i. We can, therefore, restrict attention to the belief distribution generated by the experiment, and denote experiments by the probability measures H_G and H_B that both types generate on the space of beliefs $[0,1]$. Then, for any measurable set $M \subseteq [0,1]$, $H_t(M)$ is the probability that type $t \in \{G, B\}$ generates beliefs in M.

Falsification. The agent has access to a falsification technology which enables type t items to generate signals according to $H_{\neg t}$ instead of H_t. After the principal announces a test, the agent chooses the proportion[11] p_t of type t items to disguise as $\neg t$. A falsification strategy is therefore a pair $(p_G, p_B) \in [0,1]^2$.

For example, if the agent is a car manufacturer, and an item is a car model, the agent may equip its polluting models with a device that artificially lowers emissions when the vehicle is submitted to a test. In another example, if the agent is a teacher, and items are students who must take a standardized test, he may choose to teach the test to some of his bad students. While it is natural to expect that only bad types are disguised as good types, we do not preclude good types from being disguised as bad types as part of the technology. However, we later show that it is never optimal for the agent to do so. Figure 2 depicts the effect of falsification on the interpretation of test-generated signals.

Timing. First, the principal chooses a test. Second, the agent chooses her falsification rates p_G and p_B. Third, the type (state) of each item is realized. Fourth, each item i is subjected to the test and generates a stochastic signal σ^i. Fifth, the decision maker observes the realized signals $\left\{\sigma^i\right\}_{i\in[0,1]}$, forms a belief μ^i about each item i, and takes an approval decision for each of them.

Remark 1 (Ex-ante versus interim falsification)**.** Under the continuum and independence assumptions, the law of large numbers makes it irrelevant whether the agent chooses her falsification strategy before or after observing the realized types of her items. In both cases, we can view the objective of the agent as maximizing the ex ante probability that an item is approved.

[11] Alternatively, given the continuum specification, one could think of p_t as the probability that each item of type t is disguised as type $\neg t$.

Solution Concept. As in Kamenica and Gentzkow (2011), our equilibrium concept is subgame perfect equilibrium. We often single out the choice of the test by the principal, and call it the optimal design problem, with the understanding that it is made under the assumption that other players then play according to equilibrium behavior.

Working Assumptions. In the first part of the paper, we derive the optimal test for the principal under two auxiliary assumptions. These assumptions allow us to capture the relevant trade-off in a simple way, and to focus on the main technical issues that falsification adds to the test design problem. In Section 9, we relax both assumptions and show that the optimal test we derived is still optimal.

Assumption 1 (Perfect Observability)**.** *The falsification rates p_B and p_G are observed by the decision maker before she makes her approval decisions.*

Assumption 2 (Falsification Rates Bound)**.** *The agent is restricted to falsification rates such that $p_B + p_G \leq 1$*

Under Assumption 1, because the decision maker can observe falsification rates, she updates her beliefs accordingly on and off-path. Hence, with falsification, the signal μ generated by the test can no longer be equated to the belief formed by the decision maker. A test (H_G, H_B) together with the agent's falsification rates (p_G, p_B) generate a distribution of posterior beliefs of the decision maker through Bayesian updating. In other words, the falsification rates and the test jointly generate a new Blackwell experiment. We call this distribution of beliefs an *information structure* and denote it by F. By the law of large numbers, this distribution is also the realized cross-sectional distribution of beliefs generated by the different items.

When Assumption 2 is satisfied, higher signals correspond to higher true beliefs. If the agent could choose falsification rates that do not satisfy Assumption 2, this would lead to a reversal of the meaning of signals as higher signals would lead to lower beliefs. This assumption is important under Assumption 1, as the optimal test we derive in the first part of the paper under Assumption 1 and Assumption 2 will not be immune to deviations such that $p_B + p_G > 1$ (see Appendix B). However, it is irrelevant in the true model, where we relax Assumption 1 as imperfect observability ensures that such deviations can be discouraged. We elaborate on this in Section 9. Next, we make several comments about the model that help clarify the role of these assumptions, and the consequences of our modelling choices.

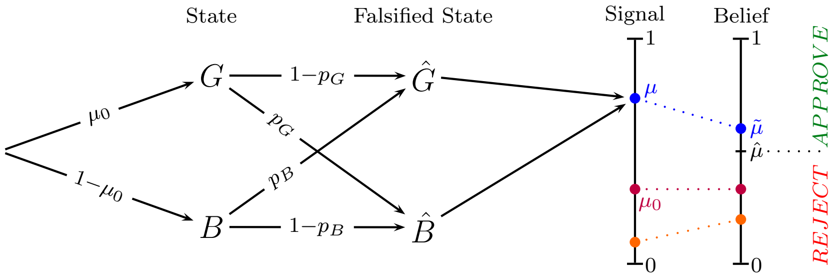


Figure 2: *The effect of falsification on beliefs under Assumption 1 and Assumption 2.*

Discussion of the Model. First, note that Assumption 1 is not necessary for the decision maker to form correct beliefs on the equilibrium path. Its importance is in allowing the decision maker to punish the agent's deviations by correctly updating beliefs off the equilibrium path. In fact, the continuum assumption implies that the decision maker can partially infer the falsification strategy of the agent by looking at the cross-sectional distribution of signals. This is the reason why we can relax Assumption 1 in Section 9. As it turns out, the fact that this inference can only be partial helps the decision maker, which is why we can also relax Assumption 2. A preview of the intuition is as follows: using the cross-sectional distribution of signals pins down cheating strategies to satisfying a certain linear equation. Adding cheating cost, or a lexicographic preference for minimal cheating, implies that only the lowest pair of cheating rates satisfying this equation can be chosen, and such pairs satisfy Assumption 2.

Second, we comment on the importance of observability, whether perfect, by Assumption 1, or partial, as granted by the continuum of items. As a benchmark, one can consider the case where falsification rates are not directly observable, and the decision maker is unable to infer them from the signal distribution. Then our problem can be formulated as a traditional mediation problem,[12] where the principal is a mediator taking reports from the agent, and making recommendations to the decision maker. In this case, it is easy to see that the mediator cannot generate any information. Indeed, to make truthful reporting by the agent incentive compatible, she must recommend approval with the same probability for good and bad items, therefore she cannot convey any information to the decision maker, and her recommendation must be to always reject since $\mu_0 < \hat{\mu}$.

Our third comment is that falsification can only make the principal less informed, in a

[12] See Myerson (1991, Chapter 6).

Blackwell sense, but does not make every garble of the test attainable. For example, the falsification technology allows the agent to render *any* test uninformative by choosing $p_B + p_G = 1$. If $\mu_0 \geq \hat{\mu}$, so that the principal approves when her belief is equal to the prior, making the test uninformative is actually the optimal choice of the agent, and there is nothing the principal can do about it. This is why, in what follows, we focus on the interesting case where $\mu_0 < \hat{\mu}$. For a given test, however, the agent cannot generate all the information structures that are less Blackwell informative than this test. This limitation is what makes the test design problem interesting. Indeed, if the agent could generate any such garbling, then the optimal design problem would always result in the optimal information structure of the agent.

Our fourth comment is on alternative choices of falsification technologies. The reason we picked this technology is because it is natural and fits well a number of examples mentioned in the introduction. However, other choices might be interesting as well. As noted above, our choice of technology limits the ways in which the agent can garble the test designed by the principal. Presumably, any choice of falsification technology would specify the ways in which tests can be garbled and the cost of doing so. If no restrictions were put on available garbles, the optimal test design problem would be moot as it would always result in the agent-optimal information structure, that is the solution of the Bayesian persuasion problem (Kamenica and Gentzkow, 2011) where the agent is the sender. This is because any test that is more informative than the agent-optimal one would be garbled back to it, whereas any other test would result in an even worse information structure for the decision maker.

Because too much falsification leads the decision maker to beliefs that punish the agent by lowering approval rates, costs are not needed to create a trade-off for the agent that the principal can exploit. And studying the problem without costs allows us to understand the effect of this trade-off more purely. Interestingly, we find that the absence of costs does not lead the agent to make the test completely uninformative when $\mu_0 < \hat{\mu}$. However, a natural extension of our falsification technology is to make it costly. Indeed, costs can capture inherent technological costs, as well as expected fines that a cheating agent may have to pay if caught, and/or ethical and emotional discomfort. We study costly falsification in Section 8.

Finally, our fifth comment is on the lack of commitment assumption by the decision maker in our model. Indeed, with commitment and (perfect or partial) observability, it is possible to generate perfect information by committing to rejecting all items whenever some cheating is observed. Such commitment is often problematic: In reality, employers, consumers, investors see test scores and decide which workers to hire, which assets to buy and so on. In the case of a single decision maker–a regulator, for example–lack of commitment captures the need for the decision maker to provide justifications, whether legal or internal, for decisions. Justifying strong punishments ('reject all') when there is a suspicion of cheating may require a higher standard of proof than mere variations in the cross-sectional grade distribution.[13]

4 Examples and Benchmarking

Binary Tests. The principal would like the decision maker to be perfectly informed about the type of the agent. But if she chooses her test to be fully informative, the agent has an incentive to falsify. In fact, faced with a fully informative test, the agent finds herself in the shoes of the sender in the Bayesian persuasion model of Kamenica and Gentzkow (2011). She chooses $p_G = 0$ and $p_B = \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}$, so that, when the decision maker sees signal $\mu = 1$, the belief she forms is exactly equal to $\hat{\mu}$. We refer to the resulting information structure as the KG information structure, and to the associated payoffs as the KG payoffs. The agent's KG payoff is $\mu_0 + (1-\mu_0)p_B = \frac{\mu_0}{\hat{\mu}}$, which is the highest possible payoff she can obtain, whereas the principal's and the decision maker's KG payoff are both 0, the payoff they would get in the absence of testing.

In many information acquisition/transmission frameworks in which the action is binary, a revelation-principle result holds which says that one can, without loss of generality, restrict attention to binary experiments. This is not the case here, but it is interesting to consider what happens with binary signals. In fact, whenever the principal chooses a binary test that is more informative than the KG information structure, the agent falsifies so as to garble it into the KG information structure. Indeed, such a test generates two signals: a low signal $\mu = 0$, and a high signal $\overline{\mu}$ above the threshold $\hat{\mu}$, where a good type generates the high signal $\overline{\mu}$ with probability 1, and a bad type generates $\overline{\mu}$ with probability $\pi_B < \mu_0 \frac{1-\hat{\mu}}{1-\mu_0}$. But then the agent

[13] In the current analysis we can incorporate such punishments in the form of cheating cost for the agent as we do in Section 8. For example, suppose that, when the agent is proved to have cheated, which happens with increasing probability $\lambda(p_B, p_G)$, he is subjected to a fine F and a recall of all his approved items. Then we can write the payoff of the agent, up to a monotonic transformation, as $\pi(p_B, p_G) - \frac{\lambda(p_B,p_G)F}{1-\lambda(p_B,p_G)}$, where $\pi(p_B, p_G)$ denotes the rate of approved items in our framework with no falsification costs. Then, solving this problem is the same as solving our model with a particular falsification cost $c(p_B, p_G)$.

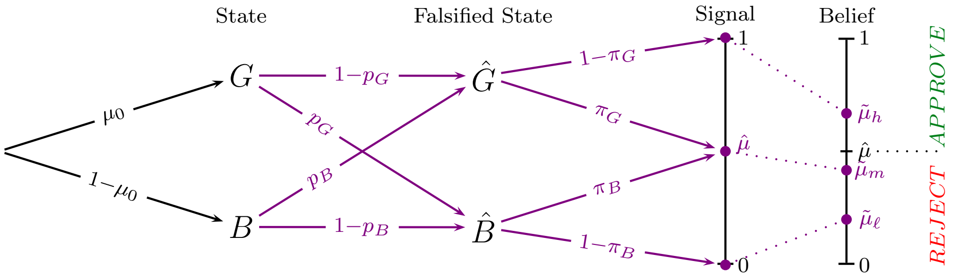


Figure 3: *A Better Test. The signal column corresponds to beliefs in the absence of falsification, the belief column gives the belief associated with each signal when there is falsification.*

obtains her KG payoff by choosing p_B so as to make the probability that a bad type generates the high signal $p_B + (1-p_B)\pi_B$ equal to $\mu_0 \frac{1-\hat{\mu}}{1-\mu_0}$, that is $p_B = \frac{1}{1-\pi_B}\left(\mu_0 \frac{1-\hat{\mu}}{1-\mu_0} - \pi_B\right)$. Hence, the principal and the decision maker get a payoff of 0. If instead the principal chooses a binary test that is less informative than, or not comparable with the KG information structure, she lowers the payoff of the agent below her KG payoff, but without increasing her own payoff. Thus, we have proved the following result.

Proposition 1 (Binary Tests)**.** *With binary tests, the principal and the decision maker always get a payoff of 0. If the test chosen by the principal is more informative than the KG information structure, the agent gets her KG payoff. Otherwise, the payoff of the agent is strictly below her KG payoff.*

A Better Test. Consider the test described in Figure 3, and recall that signals correspond to beliefs in the absence of falsification. This test has high signal generated only by G, so this signal is equal to 1, a low signal only generated by B, so it is equal to 0, and a middle signal generated by both G and B, with respective probabilities π_G and π_B, that we chose equal to $\hat{\mu}$. We pick $\pi_G = \frac{(1-\mu_0)\hat{\mu}}{\mu_0(1-\hat{\mu})}\pi_B > \pi_B$, so that the belief corresponding to the middle signal in the absence of falsification is indeed equal to $\hat{\mu}$. When the agent falsifies, the decision maker associates new beliefs to each of the three signals. These beliefs are

$$\tilde{\mu}_h = \frac{\mu_0(1-p_G)}{\mu_0(1-p_G) + (1-\mu_0)p_B},$$

$$\tilde{\mu}_m = \frac{\mu_0\pi_G - \mu_0(\pi_G - \pi_B)p_G}{\mu_0\pi_G + (1-\mu_0)\pi_B - \mu_0(\pi_G - \pi_B)p_G + (1-\mu_0)(\pi_G - \pi_B)p_B)},$$

$$\tilde{\mu}_\ell = \frac{\mu_0 p_G}{\mu_0 p_G + (1-\mu_0)(1-p_B)}.$$

Simple calculations show that $\tilde{\mu}_h$, and, more importantly, $\tilde{\mu}_m$, are decreasing in both p_G and p_B, whereas $\tilde{\mu}_\ell$ is increasing in both. Therefore any small amount of falsification implies that the agent is no longer approved when the decision maker receives the middle signal $\hat{\mu}$, as the corresponding belief falls below $\hat{\mu}$. The only benefit from falsification is therefore to increase the probability that a bad type generates the high signal by increasing p_B. Increasing p_G, however, is only harmful, so the agent should set $p_G = 0$. The maximum and optimal level of p_B is the one that brings $\tilde{\mu}_h$ down to $\hat{\mu}$, since falsifying more than this would lead the decision maker to approve none of the items. Let $\overline{p}_B = \frac{\mu_0(1-\hat{\mu})}{(1-\mu_0)\hat{\mu}}$ denote this level. The payoff of the agent if she chooses this maximum falsification level $\overline{p}_B$ is

$$\big(\mu_0 + (1-\mu_0)\overline{p}_B\big)(1-\pi_G) = \frac{\mu_0}{\hat{\mu}} - \frac{1-\mu_0}{1-\hat{\mu}}\pi_B,$$

while her no-falsification payoff is

$$\mu_0 + (1-\mu_0)\pi_B.$$

The principal can discourage falsification by equating the two, which is achieved by choosing $\pi_B^* = \frac{\mu_0(1-\hat{\mu})^2}{(1-\mu_0)\hat{\mu}(2-\hat{\mu})}$, and $\pi_G^* = \frac{1-\hat{\mu}}{2-\hat{\mu}}$. This experiment gives the principal a payoff of

$$\mu_0 g - (1-\mu_0)\pi_B^* b = (g+b)\frac{\mu_0(1-\hat{\mu})}{2-\hat{\mu}} > 0.$$

These observations are summarized in the following:

Proposition 2. *The experiment described in Figure 3 with π_B^* and π_G^* gives the agent no incentive to falsify, and yields a strictly positive payoff for the principal and the decision maker.*

Intuitively, enriching the set of signals by adding a middle signal $\hat{\mu}$ makes the agent unwilling to falsify, as any falsification would lead the decision maker to devalue the middle signal, and no longer approve items that generate this signal. This experiment, while not perfectly informative, allows the principal to generate useful information despite the possibility of costless falsification. Hence, the curse of falsification can be beaten by a good design.

We can think of several testing procedures that would generate this information structure. One is to use a perfectly informative test, and simply garble the results provided to the decision maker. Another possibility is to design two pass-fail tests to which items would be randomly and independently assigned: the first pass-fail test, assigned with probability $1-\pi^*_G$, is perfectly informative about the type, and the other one, assigned with probability π^*_G, is such that the good type passes with probability one, and the bad type with probability π^*_B/π^*_G, so that a pass in this state leads to belief $\hat{\mu}$. In this implementation, the effect of cheating is to lead the decision maker to reject all items subjected to the second test, regardless of the outcome. In the remainder of the paper, we proceed to find an optimal test.

5 Tests and Information Structures

To proceed with the general analysis, we employ a useful representation of experiments as convex functions that, to our knowledge, first appears in Kolotilin (2016), and is also discussed at length in Gentzkow and Kamenica (2016b).

Bayesian Consistency. If we denote by F both a probability measure on $[0,1]$ and the corresponding pseudo cdf,[14] it is a posterior belief distribution if and only if $\int_0^1 \mu F(d\mu) = \mu_0$ (see Kamenica and Gentzkow, 2011) or, equivalently, integrating by parts,

$$\int_0^1 F(\mu) d\mu = 1 - \mu_0. \tag{BC}$$

Experiments as Convex Functions. For a belief distribution F that satisfies (BC), we can define the function

$$\mathcal{F}(\mu) = \int_0^\mu F(x) dx$$

from $[0,1]$ to $[0, 1-\mu_0]$. Let Δ^B be the set of increasing convex functions of μ on $[0,1]$ that are bounded above by $(1-\mu_0)\mu$, and below by $(\mu-\mu_0)^+$. This set is illustrated in Figure 4.

[14] If F is a probability measure on the space of beliefs $[0,1]$, then it has a cumulative distribution function $\tilde{F} : [0,1] \to [0,1]$. Slightly abusing notations, we then denote the pseudo cdf of a probability measure F by the same letter F, and define it for $\mu \in (0,1]$ by $F(\mu) = \sup_{x<\mu} \tilde{F}(x)$. Hence, for $\mu > 0$, $F(\mu)$ is the probability measure of the set $[0,\mu)$. For example, in a perfectly informative information structure, a good item generates belief 1 with probability 1, and the bad type generates belief 0 with probability 1, that is $F_G(\mu) = 0$ and $F_B(\mu) = 1$ for all $\mu \in (0,1]$. In a perfectly uninformative experiment, both types generate belief μ_0 with probability 1, that is $F_G(\mu) = F_B(\mu) = \mathbb{1}_{\mu>\mu_0}$.

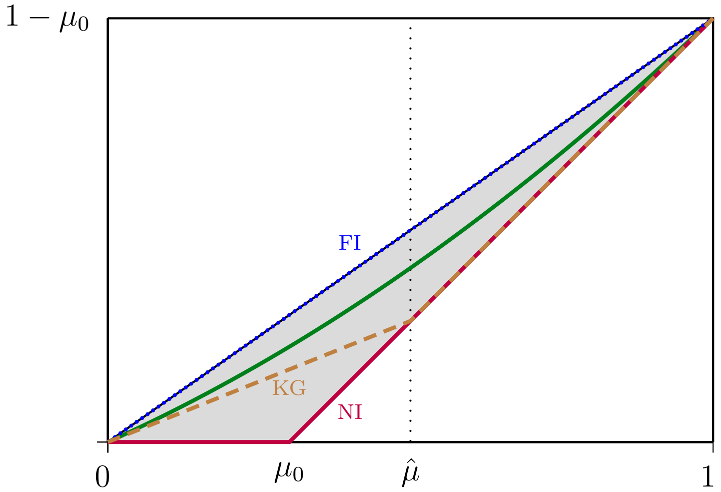


Figure 4: Δ^B *is the set of increasing convex functions in the grey triangle– the green curve is an example of a function in* Δ^B*, the brown dashed kinked line corresponds to the KG information structure which obtains when the principal uses a fully informative experiment, the top dotted blue line corresponds to full information (FI), the bottom kinked line corresponds to no information (NI). In this and all subsequent figures, we take* $\mu_0 = 0.3$ *and* $\hat{\mu} = 0.5$*.*

Then $\mathcal{F}(\cdot) \in \Delta^B$. Reciprocally, any function $\mathcal{F} \in \Delta^B$ admits a left derivative that is the pseudo cdf of a Bayes consistent belief distribution. Therefore, there is a one-to-one relationship between functions in Δ^B and Bayes consistent belief distributions. The upper bound on Δ^B corresponds to the pseudo cdf $F(\mu) = 1$, which is the fully informative experiment. The lower bound on Δ^B corresponds to the pseudo cdf $F(\mu) = \mathbb{1}_{\mu > \mu_0}$, which corresponds to the uninformative experiment and puts probability one on the prior μ_0. The following lemma states this characterization, and is proved in Appendix A.

Lemma 1. $\mathcal{F} \in \Delta^B$ *if and only if there exists a Bayes consistent belief distribution* F *such that, for all* $\mu \in [0,1]$, $\mathcal{F}(\mu) = \int_0^\mu F(x)dx$.

We can re-express the distributions of beliefs induced by good and bad types as functions of the posterior belief distribution F.

Lemma 2. *The belief distributions generated by the good type and the bad type are respectively*

$$F_G(\mu) = \frac{1}{\mu_0}\Big\{\mu F(\mu) - \mathcal{F}(\mu)\Big\},$$

$$F_B(\mu) = \frac{1}{1-\mu_0}\Big\{(1-\mu)F(\mu) + \mathcal{F}(\mu)\Big\}.$$

In the absence of falsification a test H induces an information structure, and thus satisfies Lemma 2 with the representation $\mathcal{H}$. In the presence of falsification, the test H still satisfies these relationships, that is, we have, for each signal $\mu \in (0,1]$,

$$H_G(\mu) = \frac{1}{\mu_0}\Big\{\mu H(\mu) - \mathcal{H}(\mu)\Big\},$$

and

$$H_B(\mu) = \frac{1}{1-\mu_0}\Big\{(1-\mu)H(\mu) + \mathcal{H}(\mu)\Big\}.$$

However, as already explained, the signals generated by $\mathcal{H}$ are no longer beliefs when there is falsification.

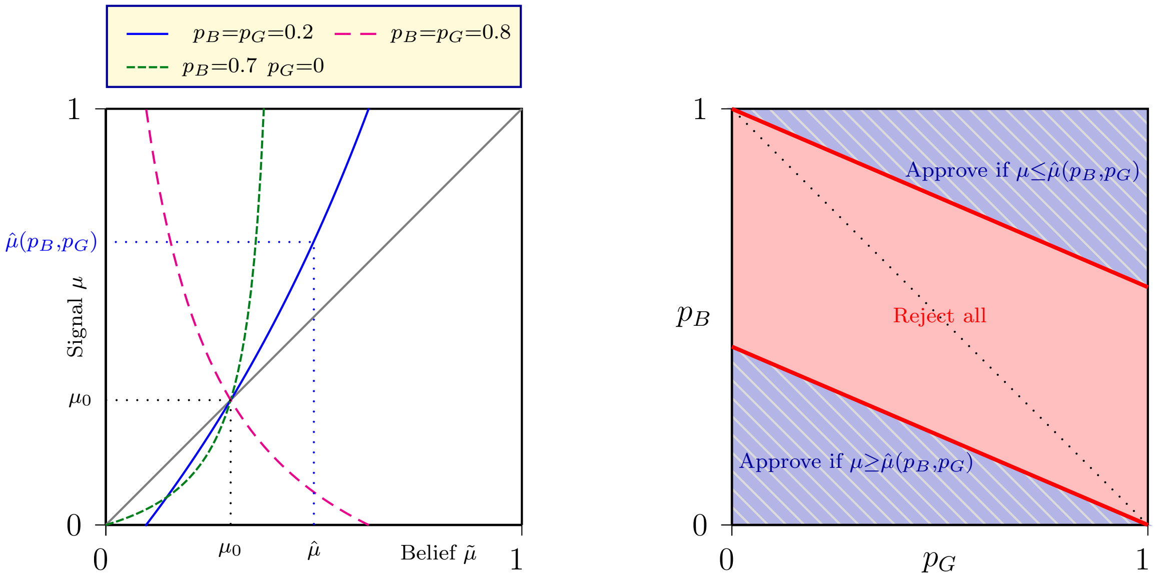
Modified Payoffs. We can obtain convenient expressions of the players' payoffs using $\mathcal{F}$. The payoff of the agent is given by the probability that she generates a belief above the threshold, $1 - F(\hat{\mu})$. Graphically, the agent would like the left derivative $F(\hat{\mu})$ of $\mathcal{F}$ at $\hat{\mu}$ to be as small as possible. The payoff of the principal, scaled by $\frac{1}{g+b}$, is

$$\begin{aligned}\frac{1}{g+b}\int_{\hat{\mu}}^{1}\big(\mu g + (1-\mu)(-b)\big)F(d\mu) &= 1 - \hat{\mu} - \int_{\hat{\mu}}^{1} F(x)dx \\ &= \mu_0 - \hat{\mu} + \mathcal{F}(\hat{\mu}).\end{aligned}$$

Since the constant terms are irrelevant for optimization, we use $\mathcal{F}(\hat{\mu})$ as our objective function for the principal. This objective function is easily pictured in Figure 4, and it appears clearly that, in the absence of any falsification constraints, the principal would choose the upper-bound function of Δ^B, which corresponds to full information (FI). It is easy to see on Figure 4 why the KG information structure is optimal for the agent, and pessimal for the principal, whereas full information is optimal for the principal. No information (NI) is pessimal for both. The payoff space generated by all possible information structures is illustrated on Figure 11, below.

6 Optimal Approval and Optimal Falsification

Optimal Approval. To understand the incentives of the agent to falsify, we start by describing how falsification affects the decision maker's approval decisions. If the agent decides



(a) The belief transformation

(b) Optimal approval policy

Figure 5: *Panel (a) illustrates the relationship between signal (or pre-falsification belief), and actual (post-falsification) belief. Panel (b) illustrates the optimal approval policy: the red line is the line with equation* $p_B = \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}(1-p_G)$*; in the solid pink region above the red line, the decision maker never approves; in the hatched blue region below the red line, she uses an approval threshold* $\hat{\mu}(p_B, p_G)$.

to falsify, he changes the belief associated with each signal. Let μ be both the signal received by the decision maker, and the belief she forms in the absence of falsification. Then, if the agent chooses a falsification strategy (p_B, p_G), the decision maker forms belief $\tilde{\mu} \neq \mu$ when she receives signal μ. Their relationship, which we call the *belief transformation*, is explicited in the next lemma, which holds for *all* values of p_B and p_G, that is, even without the restriction of Assumption 2. Interestingly, the belief transformation is independent of the test chosen by the principal, and depends only on the falsification strategy. Hence, any falsification strategy induces a reinterpretation of signals that does not depend on the test chosen by the principal.

Lemma 3 (Belief Transformation)**.** *Under Assumption 1, with falsification* (p_B, p_G)*, signal* μ *induces belief* $\tilde{\mu}$*, where*

$$\mu = \mu_0 \frac{(1-\mu_0)\tilde{\mu} - \mu_0(1-\tilde{\mu})p_G - (1-\mu_0)\tilde{\mu}p_B}{\mu_0(1-\mu_0) - \mu_0(1-\tilde{\mu})p_G - (1-\mu_0)\tilde{\mu}p_B}. \tag{BT}$$

This function has a fixed point μ_0*. It is increasing in* $\tilde{\mu}$ *if* $p_B + p_G < 1$*, decreasing if* $p_B + p_G > 1$*, and constant to* μ_0 *otherwise. The range of beliefs* $\tilde{\mu}$ *is the interval* $\left[\underline{\mu}, \overline{\mu}\right]$*, where*

$\underline{\mu} = \frac{\mu_0 p_G}{\mu_0 p_G + (1-\mu_0)(1-p_B)}$, *and* $\overline{\mu} = \frac{\mu_0(1-p_G)}{\mu_0(1-p_G)+(1-\mu_0)p_B}$.

If the amount of falsification is constrained by Assumption 2, the decision maker still associates higher signals μ with higher beliefs $\tilde{\mu}$, but this is reversed when $p_B + p_G > 1$. The belief transformation is illustrated in panel (a) of Figure 5 for different values of p_B and p_G. Note that, with falsification, beliefs may be bounded away from 0 or 1. Whenever $p_B > 0$, the decision maker can never be sure that she is facing a bad type, and whenever $p_G > 0$, she can never be sure that she is facing a good type.

The decision maker approves when her belief exceeds $\hat{\mu}$, that is when her signal μ exceeds the threshold $\hat{\mu}(p_B, p_G)$ obtained from the belief transformation, as illustrated by the first curve of panel (a) in Figure 5. For some values of (p_B, p_G), such signals cannot be generated (this is the case when $\overline{\mu} < \hat{\mu}$), and the decision maker never approves, as illustrated by the second curve of panel (a) in Figure 5. The following proposition characterizes the optimal approval strategy under falsification.

Proposition 3 (Optimal Approval)**.** *Under Assumption 1, there exists a threshold*

$$\hat{\mu}(p_B, p_G) = \mu_0 \frac{(1-\mu_0)\hat{\mu} - \mu_0(1-\hat{\mu})p_G - (1-\mu_0)\hat{\mu}p_B}{\mu_0(1-\mu_0) - \mu_0(1-\hat{\mu})p_G - (1-\mu_0)\hat{\mu}p_B},$$

such that:

(i) If $p_B < \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}(1-p_G)$*,* $\hat{\mu}(p_B, p_G)$ *is increasing in* p_B *and* p_G*, and the decision maker approves any item generating a signal* $\mu \geq \hat{\mu}(p_B, p_G)$*.*

(ii) If $p_B > 1 - \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}p_G$*,* $\hat{\mu}(p_B, p_G)$ *is decreasing in* p_B *and* p_G*, and the decision maker approves any item generating a signal* $\mu \leq \hat{\mu}(p_B, p_G)$*.*

(iii) Otherwise, the decision maker rejects every item.

The optimal policy is illustrated in panel (b) of Figure 5. Note that $\hat{\mu}(0,0) = \hat{\mu}$ as, then, signals coincide with beliefs.

Optimal Falsification. Now, consider the problem of the agent under both assumptions. Whenever there is falsification, the threshold $\hat{\mu}(p_B, p_G)$ is higher than $\hat{\mu}$. Since the threshold is increasing in p_B and p_G, more falsification hurts both types as it makes the decision maker

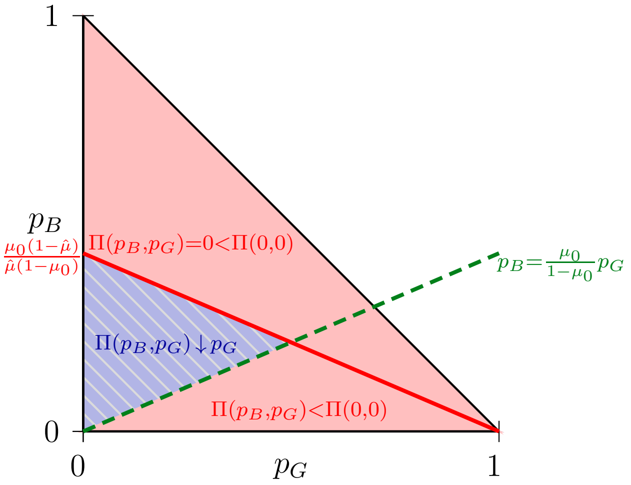

1
p_B
$\frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}$
$\Pi(p_B,p_G)=0<\Pi(0,0)$
$p_B=\frac{\mu_0}{1-\mu_0}p_G$
$\Pi(p_B,p_G)\downarrow p_G$
$\Pi(p_B,p_G)<\Pi(0,0)$
0
0
p_G
1

Figure 6: *Optimal falsification under Assumption 1 and Assumption 2 if $H(\hat{\mu}) < 1$.*

more selective. However, it also changes the probabilities with which both types generate the different signals in a way that can benefit the agent. To see this, we compute the falsification payoff of the agent. This payoff is 0 in the region where the decision maker rejects for all signals. In the threshold region, we can write the agent's payoff as

$$\Pi(p_B,p_G) = 1-\big\{\mu_0(1-p_G)+(1-\mu_0)p_B\big\}H_G\big(\hat{\mu}(p_B,p_G)\big)-\big\{\mu_0 p_G+(1-\mu_0)(1-p_B)\big\}H_B\big(\hat{\mu}(p_B,p_G)\big).$$

Using the expressions from Lemma 2 applied to H_G and H_B, we obtain

$$\Pi(p_B,p_G) = 1-H\big(\hat{\mu}(p_B,p_G)\big)+\left(\frac{p_B}{\mu_0}-\frac{p_G}{1-\mu_0}\right)\Big\{\mathcal{H}\big(\hat{\mu}(p_B,p_G)\big)-\big(\hat{\mu}(p_B,p_G)-\mu_0\big)H\big(\hat{\mu}(p_B,p_G)\big)\Big\}. \tag{1}$$

This expression, as we show, implies that, in any optimal falsification strategy that follows a relevant test, $p_G = 0$. Intuitively, pretending that items are bad when in fact they are good not only increases the approval threshold, but also deteriorates the signal distribution generated by good types. It may, however, be payoff-improving for the agent to sometimes pretend that an item is good when in fact it is bad. Even though it increases the approval threshold, it allows bad items to generate the same signal distribution as good ones, and therefore be approved with a higher probability.

Proposition 4 (Optimal Falsification)**.** *Under Assumption 1, and Assumption 2, any optimal* *falsification strategy satisfies the following.*

(i) If $H(\hat{\mu}) < 1$, then $p_G = 0$ and $p_B \leq \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}$.

(ii) If $H(\hat{\mu}) = 1$, then falsification is inconsequential, the decision maker never approves, and all players get a null payoff.

The idea of the proof, can be visualized on Figure 6. First, we show that all falsification strategies that do not lie in the hatched triangle are dominated by no falsification. Second, we show that $\Pi(p_B, p_G)$ is decreasing in p_G within the hatched triangle.

Proposition 4 implies that the optimal falsification problem of the agent can be reduced to the choice of $p_B \in \left[0, \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}\right]$, thus generating an approval threshold $\hat{\mu}(p_B, 0)$ between $\hat{\mu}$ and 1. We can reformulate this problem as the choice of a threshold $\mu \in [\hat{\mu}, 1]$, and invert the function $\hat{\mu}(p_B, 0)$ to get the level of falsification p_B that corresponds to a threshold μ,

$$p_B = \frac{\mu_0(\mu - \hat{\mu})}{\hat{\mu}(\mu - \mu_0)}.$$

Replacing this in (1), we obtain the falsification payoff of the agent as a function of the induced signal threshold μ

$$\begin{aligned}\Pi(\mu) &= 1 - H(\mu) + \frac{\mu - \hat{\mu}}{\hat{\mu}(\mu - \mu_0)}\Big\{\mathcal{H}(\mu) - (\mu - \mu_0)H(\mu)\Big\} \\ &= 1 + \frac{\mu - \hat{\mu}}{\hat{\mu}(\mu - \mu_0)}\mathcal{H}(\mu) - \frac{\mu}{\hat{\mu}}H(\mu), \qquad (2)\end{aligned}$$

and the agent's optimal falsification problem reduces to choosing which approval threshold to induce so as to maximize $\Pi(\mu)$ on $[\hat{\mu}, 1]$.

7 Optimal Design

We now consider the optimal test design problem of the principal in the presence of falsification, under Assumption 1 and Assumption 2. Both these assumptions are relaxed in Section 9.

A No-Falsification Principle. We start by showing that a no-falsification principle holds. It states that any final information structure and, therefore, any payoffs the principal can generate with falsification, can also be generated without falsification. The logic of the argument is similar to that of the revelation principle. Consider any test, and the optimal falsification strategy of the agent associated with this test. Together, they generate a certain information structure. Now, consider providing the agent with the test that generates this precise information structure, instead of the initial test. Then, we show that the agent has no incentive to falsify under this new test. The main difference with the usual revelation principle is in the link between deviations from no falsification under the new test, and corresponding deviations from the optimal level of falsification under the initial test.

More formally, suppose that the principal chooses a test H, and let $p_B^* > 0$ be the associated optimal falsification strategy of the agent. Together, p_B^* and H define a new experiment, characterized by a posterior belief distribution F. One way to deliver this experiment, is to choose the test F described in the lower panel of Figure 7. As illustrated by Figure 7, falsifying by choosing $p_B = \varepsilon$ under this new test induces the same posterior belief distribution as increasing the level of falsification by $(1-p_B^*)\varepsilon$ under the initial test H. But since p_B^* is optimal under H, this deviation must be unprofitable to the agent. Therefore, it is optimal for the agent not to falsify under the new test F. This proves the no-falsification principle,[15] which we now state more formally.

Proposition 5 (No-Falsification Principle)**.** *If the principal can induce a final belief distribution F with falsification $p_B^* > 0$, then she can also induce this distribution with no falsification. In both cases, her payoff is given by $\mathcal{F}(\hat{\mu})$, and the payoff of the agent by $1 - F(\hat{\mu})$.*

Optimal Design. The no-falsification principle implies that we can restrict the optimal design problem to the one of finding an optimal test under which the agent has no incentive to falsify. A test H is such that the agent has no incentive to falsify if and only if $\Pi(\hat{\mu}) \geq \Pi(\mu)$, for all $\mu \in [\hat{\mu}, 1]$, that is, recalling the payoff formula (2), if and only if H satisfies the following incentive constraint

$$\frac{\mu - \hat{\mu}}{\mu - \mu_0}\mathcal{H}(\mu) \leq \mu H(\mu) - \hat{\mu} H(\hat{\mu}), \qquad \forall \mu \in \big[\hat{\mu}, 1\big]. \tag{IC$_0$}$$

[15] The no-falsification principle is more general than the version we state in the theorem. It holds for any state space (not just binary as in our model) so long as falsification is costless or that falsification costs are concave in falsification rates. Details are available from the authors upon request.

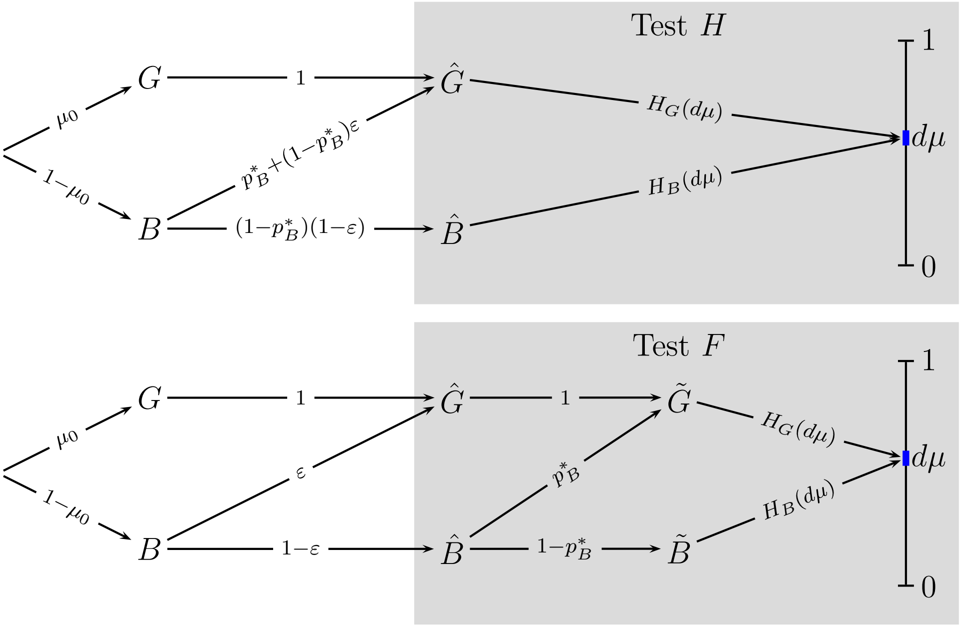


Figure 7: *Experiment and final information structure with p_B^*.*

And, if this is the case, the payoff of the principal is given by $\mathcal{H}(\hat{\mu})$ (up to constants). Hence the program of the principal is

$$\max_{\mathcal{H}\in\Delta^B} \mathcal{H}(\hat{\mu})$$
$$\text{s.t. } \frac{\mu-\hat{\mu}}{\mu-\mu_0}\mathcal{H}(\mu) \le \mu H(\mu) - \hat{\mu}H(\hat{\mu}), \qquad \forall \mu \in \big[\hat{\mu}, 1\big]. \tag{IC$_0$}$$

To form intuition about this program, it is useful to go back to Figure 4. The principal wants to maximize $\mathcal{H}(\hat{\mu})$ subject to a constraint on the values taken by $\mathcal{H}$ to the right of $\hat{\mu}$. There is no incentive constraint on $\mathcal{H}$ to the left of $\hat{\mu}$. Recall that $H(\hat{\mu})$ is the left-derivative of $\mathcal{H}$ at $\hat{\mu}$.

A first remark is that we can look for optimal tests that are linear to the left of $\hat{\mu}$. To see this, suppose that $\mathcal{H} \in \Delta^B$ satisfies (IC$_0$), and consider the function

$$\tilde{\mathcal{H}}(\mu) = \begin{cases} \mu\mathcal{H}(\hat{\mu})/\hat{\mu} & \text{if } \mu \le \hat{\mu} \\ \mathcal{H}(\mu) & \text{if } \mu \ge \hat{\mu} \end{cases}.$$

It is easy to see that $\tilde{\mathcal{H}}$ is in Δ^B, and since $\tilde{H}(\hat{\mu}) = \mathcal{H}(\hat{\mu})/\hat{\mu} \le H(\hat{\mu})$, by convexity of $\mathcal{H}$, the new experiment $\tilde{\mathcal{H}}$ also satisfies (IC$_0$), and delivers the same payoff to the principal. Therefore, we have proved the following lemma.

Lemma 4. *For every test $\mathcal{H}$ that satisfies* (IC$_0$)*, there is a test $\tilde{\mathcal{H}}$ that is linear to the left of $\hat{\mu}$, satisfies* (IC$_0$)*, and delivers the same payoff to the principal.*

Linearity means that we can look for optimal tests that put an atom on belief 0, and never generate any belief in $\big(0, \hat{\mu}\big)$. In particular, we can restrict ourselves to tests such that good types are never rejected. Another consequence of Lemma 4 is that we can look for optimal tests that are on the Pareto frontier. Indeed, recalling the definition of the set Δ^B, it is easy to visualize on Figure 4 that $\tilde{\mathcal{H}}$ is the test with the lowest possible left derivative at $\hat{\mu}$ among tests that deliver payoff $\mathcal{H}(\hat{\mu})$ to the principal.

Next, we denote the left derivative of $\mathcal{H}$ at $\hat{\mu}$ by κ. Since $\mathcal{H} \in \Delta^B$, we must have $0 \leq \kappa \leq 1 - \mu_0$. Note that the (IC$_0$) constraint is automatically satisfied at $\hat{\mu}$. Therefore, we can rewrite it as

$$\mu H(\mu) - \frac{\mu - \hat{\mu}}{\mu - \mu_0} \mathcal{H}(\mu) \geq \kappa \hat{\mu}, \qquad \forall \mu > \hat{\mu}. \tag{IC$'_0$}$$

Then, the principal's problem reduces to choosing $\kappa \in \big[0, 1 - \mu_0\big]$, and $\mathcal{H} \in \Delta^B$ such that $\mathcal{H}(\mu) = \kappa\mu$ for $\mu \leq \hat{\mu}$ so as to maximize κ, under the constraint (IC$'_0$).

As a first exercise, we can find the optimal test with three signals, and compare it to the test we described in Section 4. This test must be linear to the right of $\hat{\mu}$. Let η be its slope to the right of $\hat{\mu}$. We must have $\eta = \frac{1-\mu_0-\kappa\hat{\mu}}{1-\hat{\mu}}$. And we can rewrite (IC$'_0$) as

$$\eta\mu - \frac{\mu - \hat{\mu}}{\mu - \mu_0}\big(\kappa\hat{\mu} + \eta(\mu - \hat{\mu})\big) \geq \kappa\hat{\mu}, \qquad \forall \mu > \hat{\mu}.$$

A quick calculation shows that the left-hand side is strictly decreasing in μ. So the incentive constraint can be simplified to

$$\eta - \frac{1 - \hat{\mu}}{1 - \mu_0}\big(\kappa\hat{\mu} + \eta(1 - \hat{\mu})\big) \geq \kappa\hat{\mu}.$$

Replacing η by its expression, and rearranging, we obtain

$$\kappa \leq \frac{(1 - \mu_0) - (1 - \hat{\mu})^2}{\hat{\mu}(2 - \hat{\mu})}.$$

Since the principal wants to maximize $\mathcal{H}(\hat{\mu}) = \kappa\hat{\mu}$, this constraint must bind at the optimum,

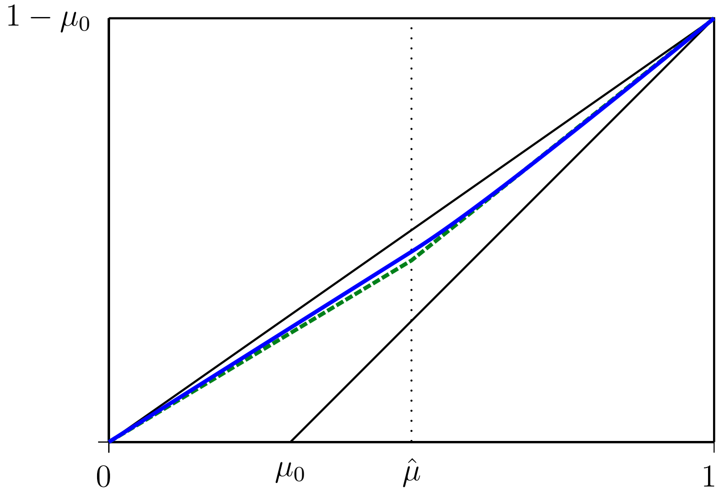


Figure 8: *Optimal Design – the lower dashed curve is the optimal three-signal test, and the higher curve is our optimal test.*

that is, the optimal choice of κ is

$$\kappa^*_{3S} = \frac{(1-\mu_0)-(1-\hat{\mu})^2}{\hat{\mu}(2-\hat{\mu})}.$$

Proposition 6. *The optimal three-signal test is*

$$\mathcal{H}^*_{3S}(\mu) = \frac{(1-\mu_0)-(1-\hat{\mu})^2}{\hat{\mu}(2-\hat{\mu})}\mu + \frac{2-\mu_0-\hat{\mu}}{2-\hat{\mu}}\big(\mu-\hat{\mu}\big)^+,$$

and it corresponds to the one described in Proposition 2.

This experiment is illustrated in Figure 8, which also depicts the optimal test that we characterize next. In order to do so, we first define the unique test that makes the agent indifferent across all falsification levels p_B that induce an approval threshold between $\hat{\mu}$ and 1. Then, we proceed to show that this test is optimal. Such a test must satisfy the incentive constraint (IC'_0) everywhere with equality, and must therefore solve the *indifference differential equation*

$$H(\mu) - \frac{\mu-\hat{\mu}}{\mu(\mu-\mu_0)}\mathcal{H}(\mu) = \frac{\kappa\hat{\mu}}{\mu}, \tag{IDE}$$

on $\left[\hat{\mu},1\right]$, with initial condition $\mathcal{H}(\hat{\mu})=\kappa\hat{\mu}$. The unique solution to this problem is given by

$$\mathcal{H}(\mu)=\kappa\hat{\mu}\psi(\mu)\left(1+\int_{\hat{\mu}}^{\mu}\frac{1}{x\psi(x)}dx\right),$$

where

$$\psi(\mu)=\exp\left(\int_{\hat{\mu}}^{\mu}\frac{x-\hat{\mu}}{x(x-\mu_0)}dx\right).$$

If $\mathcal{H}\in\Delta^B$, it must satisfy $\mathcal{H}(1)=1-\mu_0$. Adding this constraint pins down the value of κ to

$$\kappa^*=\frac{1-\mu_0}{\hat{\mu}\psi(1)\left(1+\int_{\hat{\mu}}^{1}\frac{1}{x\psi(x)}dx\right)}.$$

Theorem 1. *The test defined by*

$$\mathcal{H}^*(\mu)=\begin{cases}\kappa^*\mu & \text{if } \mu\leq\hat{\mu}\\ \kappa^*\hat{\mu}\psi(\mu)\left(1+\int_{\hat{\mu}}^{\mu}\frac{1}{x\psi(x)}dx\right) & \text{if } \mu\geq\hat{\mu}\end{cases}$$

is optimal. Furthermore, any other optimal test must be linear to the left of $\hat{\mu}$ and less informative than $\mathcal{H}^*$.

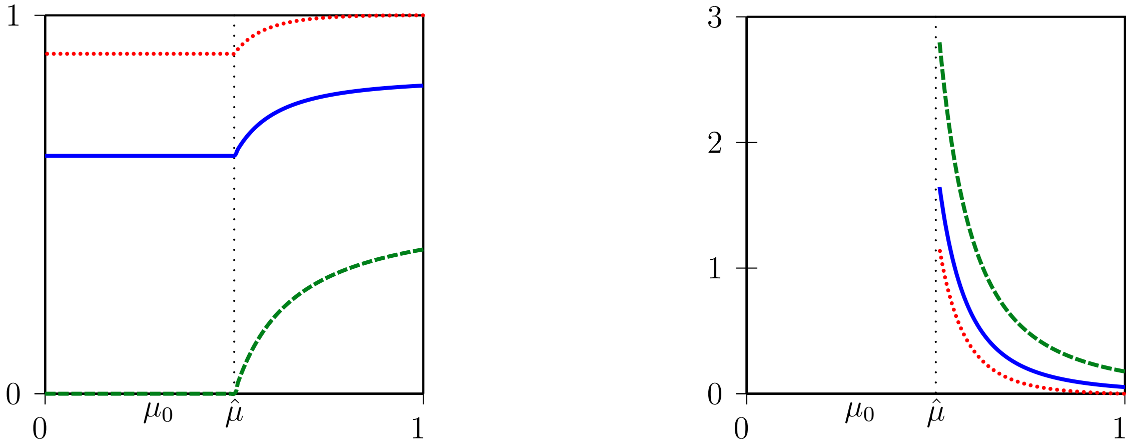
Proof. The proof consists of three steps. The first step is to show that $\mathcal{H}^*$ is indeed in Δ^B, so that it is actually a test. This purely calculatory part is proved in the appendix. The third step is to show that any other optimal experiment is linear to the left of $\hat{\mu}$, and less informative. It is relegated to the appendix as well. In what follows, we provide the second and most interesting step of the proof, which consists in showing that no incentive compatible experiment can do better than $\mathcal{H}^*$.

To see this, suppose that there exists an experiment $\mathcal{H}\in\Delta^B$ that satisfies (IC_0'), and $\mathcal{H}(\hat{\mu})>\mathcal{H}^*(\hat{\mu})$. Lemma 4 implies that we can additionally chose it to be linear to the left of $\hat{\mu}$, with slope $\kappa>\kappa^*$, as $\kappa\hat{\mu}=\mathcal{H}(\hat{\mu})>\mathcal{H}^*(\hat{\mu})=\kappa^*\hat{\mu}$. Since $\mathcal{H}(1)=\mathcal{H}^*(1)=1-\mu_0$, the intermediate value theorem applied to the difference of $\mathcal{H}-\mathcal{H}^*$, which is continuous by convexity of each of these functions, implies that $\mathcal{H}$ and $\mathcal{H}^*$ cross at least once on $\left(\hat{\mu},1\right]$. Let $\tilde{\mu}$ be the smallest of these crossing points. Then $\mathcal{H}(\mu)>\mathcal{H}^*(\mu)$ for every $\mu\in\left[\hat{\mu},\tilde{\mu}\right]$, which implies that the left-derivative of $\mathcal{H}$ at $\tilde{\mu}$ is smaller than the left derivative of $\mathcal{H}^*$ at $\tilde{\mu}$, that is

$H(\tilde{\mu}) \leq H^*(\tilde{\mu})$. Therefore, we have

$$\tilde{\mu} H(\tilde{\mu}) - \frac{\tilde{\mu} - \hat{\mu}}{\tilde{\mu} - \mu_0} \mathcal{H}(\tilde{\mu}) \leq \tilde{\mu} H^*(\tilde{\mu}) - \frac{\tilde{\mu} - \hat{\mu}}{\tilde{\mu} - \mu_0} \mathcal{H}^*(\tilde{\mu}) = \kappa^* \hat{\mu} < \kappa \hat{\mu},$$

which implies that $\mathcal{H}$ cannot satisfy (IC_0'), a contradiction. □



(a) Pseudo CDFs

(b) Densities

Figure 9: *Optimal Design – in each panel, the blue curve in the middle is the distribution of beliefs, the dashed green curve is the distribution of beliefs generated by the good type, and the dotted red curve is the distribution of beliefs generated by the bad type.*

The optimal test is illustrated in Figure 8 and Figure 9. In the proof of Theorem 1, we derive a closed form expression of the optimal test without integrals. For every $\mu \geq \hat{\mu}$,

$$\mathcal{H}^*(\mu) = \kappa^*(\mu - \mu_0) \left\{ 1 + \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} \hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}} \left(\frac{\mu}{\mu - \mu_0} \right)^{\frac{\hat{\mu}}{\mu_0}} \right\}.$$

Using this expression we establish that $\mathcal{H}^*$ satisfies the following properties:

Proposition 7. *The belief distribution generated by the optimal test has support on $\{0\} \cup [\hat{\mu}, 1]$, with atoms at 0 and 1, and a positive, continuously differentiable, and decreasing density on $[\hat{\mu}, 1)$. The belief distribution generated by the good type has support on $[\hat{\mu}, 1]$, with a positive, continuously differentiable, and decreasing density on $[\hat{\mu}, 1)$, and a single atom at 1. The belief distribution of the bad type has support on $\{0\} \cup [\hat{\mu}, 1]$, with a single atom at 0, and a positive, continuously differentiable, and decreasing density on $[\hat{\mu}, 1)$. Furthermore, the belief distribution generated by the good type first-order stochastically dominates that of the bad type.*

Hence, optimal tests use a rich set of signals. They involve a continuum of signals despite the fact that types and actions are binary. The richness of optimal tests is only in the "passing" signals as only one signal is associated with failure. Note that Figure 9 shows a clustering of grades close to the threshold. Intuitively, enriching the set of signals that lead to approval allows the principal to get better information while discouraging falsification. Increasing falsification would increase the probability that the bad type generates the continuum of signals above $\hat{\mu}$ rather than the reject signal. But the principal would react by rejecting some of the signals above $\hat{\mu}$ in an amount that exactly offsets the advantage from the first effect.

Our optimal test makes the agent indifferent across all moderate levels of falsification as it satisfies (IDE). Indifference of "the agent" at the optimal information structure also appears in Roesler and Szentes (2017) or Chassang and Ortner (2016). In our context, a test which makes no-falsification strictly better than some other falsification threshold cannot be optimal, since the principal can increase the informativeness of that test and still maintain that no falsification is a best response for the agent.

Implementation. As in the three-signal example, there are multiple ways to implement the optimal information structure. Obtaining a perfect information and then garbling it before transmitting it to the decision maker is one way. Another way is to design continuum of pass-fail tests assigned to each item randomly and independently with carefully chosen probabilities. Each of these pass-fail tests if failed only by the bad type, but can be passed by both, so that passing leads to a belief $\mu \geq \hat{\mu}$, and these beliefs index the continuum of pass-fail tests. The fully informative pass-fail test is assigned with probability $1 - H(1)$, whereas the other tests are assigned with probability $h_G(\mu)$, and are such that the good type passes with probability 1, but the bad type only with probability $h_B(\mu)/h_G(\mu)$, so that passing leads to belief μ.

Performance. We compare the performance of optimal tests and optimal three-signal tests with the full information. This comparison is meant as a simple illustration and it is depicted in Figure 10 which also gives a sense of comparative statics. Both optimal tests deliver at least 50% of the full information payoff. A numerical analysis shows that the optimal three-signal test delivers at least around 80% of the optimal test suggesting that most of the benefits can be harvested with simple tests using a small number of signals.

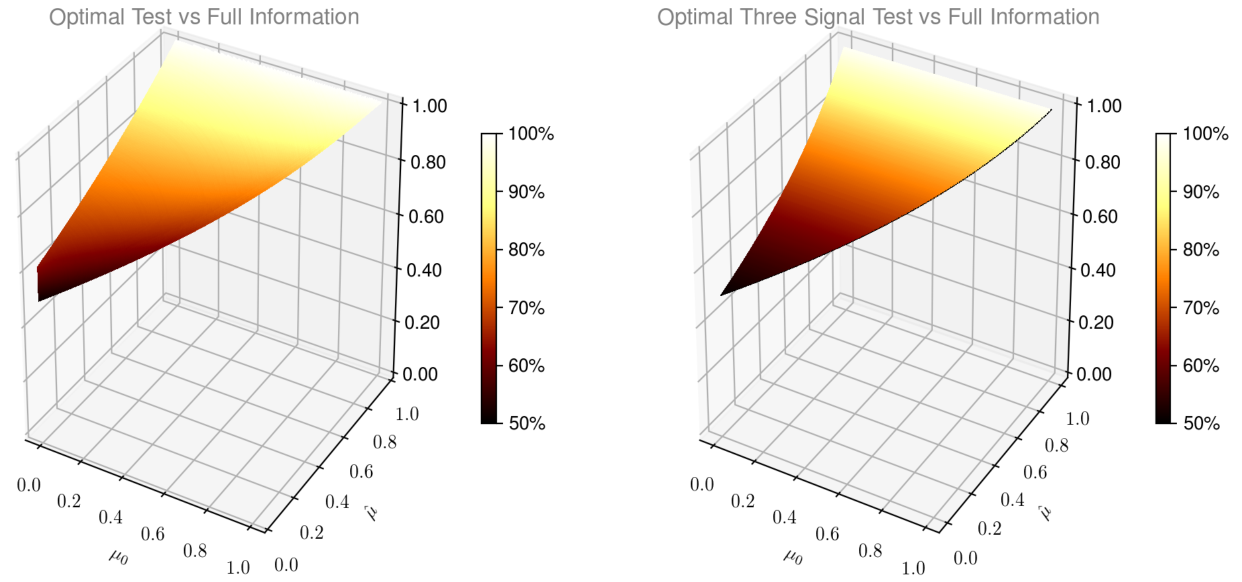


Figure 10: *Performance of $\mathcal{H}^*$ and $\mathcal{H}^*_{3S}$ in percentage of the full information payoff*

Proposition 8. *$\mathcal{H}^*$ and $\mathcal{H}^*_{3S}$ are ex-ante Pareto efficient. With both tests, the principal obtains at least 1/2 of the full information payoff. Furthermore, this bound is strict since one can find a sequence of pairs $(\mu_0, \hat{\mu})$ such that the payoff ratio gets arbitrarily close to 1/2.*

Figure 11 shows the outcome of different information structures in the payoff space, and illustrates the efficiency of both tests. The outcome is always on the Pareto frontier.

8 Costly Falsification

In this section, we study optimal test design when falsification is costly. We model this with a cost function $C(p_B, p_G) \geq 0$. The cost can be thought of as a combination of a technological scaling cost, and an expected punishment cost of being caught-which could be explicit, psychological, or reputational. We naturally assume that $C(\cdot)$ is continuous and increasing in p_B and p_G, and that $C(0,0) = 0$. The optimal approval strategy described in Proposition 3 applies to the case of costly falsification without any modifications. Then, the fact that $C(p_B, p_G)$ is increasing in p_G ensures that the optimal falsification result of Proposition 4 holds with cost, so the agent always chooses $p_G = 0$. Furthermore, the relevant range for p_B is again the interval $I = \left[0, \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}\right]$. As a consequence, to simplify notations, we can define the new cost function

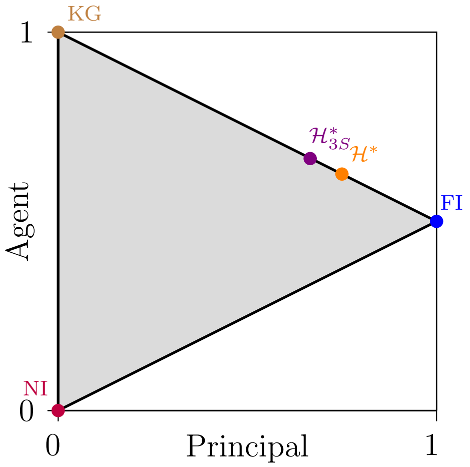


Figure 11: *Information structures in payoff space. Each player's payoff is expressed in percentage of her maximum attainable payoff. The grey triangle is the space of attainable payoffs, and the dots represent the payoffs achieved by different information structures.*

$c(p_B) = C(p_B, 0)$.

An important building block of our analysis is the no-falsification principle. In order for the principle to hold, it must be no more costly to raise falsification from any p_B^* to $p_B^* + (1 - p_B^*)\varepsilon$, than it is to raise it from 0 to ε. This is satisfied whenever $c(p_B)$ is concave in p_B, but we can also accommodate some moderately convex functions with a positive marginal cost at 0. The following assumption on the cost function ensures that the no-falsification principle holds.[16]

Assumption 3. *For every $p_B \in I$ and every $\varepsilon > 0$ such that $p_B + \varepsilon \in I$,*

$$c(\varepsilon) \geq c\big(p_B + (1 - p_B)\varepsilon\big) - c(p_B).$$

Under these assumptions, we can formulate the optimal design problem as before. The only difference is that we need to account for the cost in the no-falsification incentive constraint, which becomes

$$\frac{\mu - \hat{\mu}}{\mu - \mu_0}\mathcal{H}(\mu) - \hat{\mu} c\left(\frac{\mu_0(\mu - \hat{\mu})}{\hat{\mu}(\mu - \mu_0)}\right) \leq \mu H(\mu) - \hat{\mu} H(\hat{\mu}), \qquad \forall \mu \in \big[\hat{\mu}, 1\big]. \tag{IC$_0^c$}$$

Intuitively, costly falsification should allow the principal to attain more informative information structures. Hence, we can start by looking for conditions on the cost function that allow

[16] Note that, if $c(\cdot)$ is differentiable at 0, Assumption 3 is equivalent to requesting that $c'(0) \geq (1 - p_B)c'(p_B)$ for every $p_B \in I$ at which $c(\cdot)$ is differentiable.

the principal to attain full information. The fully informative test is given by $\mathcal{H}(\mu) = (1-\mu_0)\mu$, and is incentive compatible if, for every $\mu \in \left[\hat{\mu}, 1\right]$,

$$c\left(\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}\right) \geq (1-\mu_0)\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}.$$

That is, if the cost function satisfies the following full information condition

$$c(p_B) \geq (1-\mu_0)p_B, \qquad \forall p_B \in I. \tag{FI}$$

This also shows (replacing the inequality by an equality), that the cost function $c(p_B) = (1-\mu_0)p_B$ is the unique one that makes the agent indifferent across all the thresholds she might induce by falsifying under the fully informative test.

In what follows, we assume that $c(p_B) = \lambda p_B$, with $\lambda > 0$. Such linear cost functions lend themselves to interesting comparative static results and tractable analysis.[17] Note that Assumption 3 is automatically satisfied by linear cost functions. Moreover, $c(p_B)$ satisfies (FI) if and only if $\lambda \geq 1-\mu_0$. Otherwise, we write the indifference differential equation, which is given by

$$H(\mu) - \frac{\mu-\hat{\mu}}{\mu(\mu-\mu_0)}\mathcal{H}(\mu) = \frac{\kappa\hat{\mu}}{\mu} - \lambda\frac{\mu_0(\mu-\hat{\mu})}{\mu(\mu-\mu_0)}.$$

Its solution with initial condition $\mathcal{H}(\hat{\mu}) = \kappa\hat{\mu}$ is

$$\mathcal{H}(\mu) = \hat{\mu}\psi(\mu)\left[\kappa\left(1+\int_{\hat{\mu}}^{\mu}\frac{1}{x\psi(x)}dx\right) - \lambda\frac{\mu_0}{\hat{\mu}}\int_{\hat{\mu}}^{\mu}\frac{x-\hat{\mu}}{x(x-\mu_0)\psi(x)}dx\right],$$

and the unique value of κ that ensures that $\mathcal{H}(1) = 1-\mu_0$ is

$$\kappa^*_\lambda = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \lambda\frac{\mu_0}{\hat{\mu}}\int_{\hat{\mu}}^{1}\frac{x-\hat{\mu}}{x(x-\mu_0)\psi(x)}dx\right)\left(1+\int_{\hat{\mu}}^{1}\frac{1}{x\psi(x)}dx\right)^{-1}.$$

Then, we have the following result.

Theorem 2. *If $\lambda \geq 1-\mu_0$, then the optimal test is the fully informative one. Otherwise, the*

[17] The complete solution for arbitrary cost functions that satisfy Assumption 3 is complicated because the solution of the differential equation may not define a test. In Appendix C, we show how we can modify the cost function recursively to obtain a solution for a more general class of cost functions. In the case of a linear cost, the recursive approach is not necessary.

test given by

$$\mathcal{H}^*_\lambda(\mu) = \begin{cases} \kappa^*_\lambda \mu & \textit{if } \mu \leq \hat{\mu} \\ \hat{\mu}\psi(\mu)\left[\kappa^*_\lambda\left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx\right) - \lambda\frac{\mu_0}{\hat{\mu}}\int_{\hat{\mu}}^{\mu}\frac{x-\hat{\mu}}{x(x-\mu_0)\psi(x)}dx\right] & \textit{if } \mu \geq \hat{\mu} \end{cases}$$

is optimal. Furthermore, any other optimal experiment must be linear to the left of $\hat{\mu}$, and less informative than $\mathcal{H}^*_\lambda$. Finally, for all $\mu \in (0,1)$, $\mathcal{H}_{FI}(\mu) > \mathcal{H}^*_\lambda(\mu) > \mathcal{H}^*(\mu)$.

In the proof of Theorem 2, we derive the following expression for $\mathcal{H}^*_\lambda$. For every $\mu \geq \hat{\mu}$,

$$\mathcal{H}^*_\lambda(\mu) = \kappa^*_\lambda \mu + (\kappa^*_\lambda - \lambda)\mu_0 \left\{ \left(\frac{\mu}{\hat{\mu}}\right)^{\frac{\hat{\mu}}{\mu_0}} \left(\frac{\hat{\mu}-\mu_0}{\mu-\mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}-1} - 1 \right\}.$$

With a linear cost, the optimal test has the same qualitative properties as without cost.

Proposition 9. *Suppose $\lambda < 1-\mu_0$. Then, the belief distribution generated by our optimal test has support on $\{0\}\cup\left[\hat{\mu},1\right]$, with atoms at 0 and 1, and a positive, continuously differentiable, and decreasing density on $\left[\hat{\mu},1\right)$. The belief distribution generated by the good type has support on $\left[\hat{\mu},1\right]$, with a positive, continuously differentiable, and decreasing density on $\left[\hat{\mu},1\right)$, and a single atom at 1. The belief distribution of the bad type has support on $\{0\}\cup\left[\hat{\mu},1\right]$, with a single atom at 0, and a positive, continuously differentiable, and decreasing density on $\left[\hat{\mu},1\right)$. Furthermore, the belief distribution generated by the good type first-order stochastically dominates that of the bad type.*

In addition, we can derive the following comparative statics in λ confirming the initial intuition that higher costs lead to more informative optimal tests.

Proposition 10. *For $\lambda \leq 1-\mu_0$, the Blackwell informativeness of $\mathcal{H}^*_\lambda$ is strictly increasing in λ.*

9 Relaxing the Assumptions

In the baseline analysis, we have assumed that falsification rates are perfectly observable by the decision maker (Assumption 1), and that they must satisfy $p_B + p_G \leq 1$ (Assumption 2). The latter assumption guarantees that the meaning of grades is not flipped (higher signals are associated with a higher belief that an item is good). Interestingly, as we explain in Appendix B, the reason we need Assumption 2 is because we impose the perfect observability Assumption 1. However, perfect observability is likely to be unjustified in many contexts. We now drop both these assumptions and derive the optimal falsification-proof test.

Relaxing perfect observability and falsification limits. On the equilibrium path falsification rates are correctly anticipated even if they are unobserved. The issue arises for off-path information sets. However, the fact that the agent has a continuum of items that he subjects to testing, allows the decision maker to make inferences about the agent's falsification rates from the empirical distribution of test results:[18]

If H denotes a test chosen by the principal, then, for any choice of falsification (p_B, p_G), the cross-sectional distribution of signals observed by the decision maker is

$$\begin{aligned} F(\mu) &= \big\{\mu_0(1-p_G) + (1-\mu_0)p_B\big\}H_G(\mu) + \big\{\mu_0 p_G + (1-\mu_0)(1-p_B)\big\}H_B(\mu) \\ &= H(\mu) + \left(\frac{p_G}{1-\mu_0} - \frac{p_B}{\mu_0}\right)\big\{\mathcal{H}(\mu) - (\mu-\mu_0)H(\mu)\big\}. \end{aligned}$$

Hence, for every test that is not the uninformative test, the decision maker can compute $\frac{p_G}{1-\mu_0} - \frac{p_B}{\mu_0}$ from the cross-sectional distribution of signals. She cannot perfectly observe the choice of falsification of the agent, since she cannot tell apart two strategies (p_B, p_G) and (p'_B, p'_G) such that $\frac{p_G}{1-\mu_0} - \frac{p_B}{\mu_0} = \frac{p'_G}{1-\mu_0} - \frac{p'_B}{\mu_0}$. Therefore, the information sets of the decision maker are the sets

$$\mathcal{I}_\alpha = \left\{(p_B, p_G) \in [0,1]^2 : p_B = \frac{\mu_0}{1-\mu_0}p_G + \alpha\right\},$$

for $\alpha \in [-1, 1]$.

A strategy of the decision maker specifies an approval policy conditioned on signals for each of her information sets. Since all falsification choices (p_B, p_G) that belong to the same information set $\mathcal{I}_\alpha$ generate the same distribution of signals F, any strategy of the decision maker leads to the same approval probabilities of good and bad items for all $(p_B, p_G) \in \mathcal{I}_\alpha$.

[18] Such linking of decisions has shown to be useful by Jackson and Sonnenschein (2007) who establish that the incentive costs become negligible by constructing a mechanism in which each agent announces preferences over many decisions. These announcements must be "budgeted" such that the distribution of types across problems must mirror the underlying distribution of their preferences. Analogously, in our setup Bayes' rule implies the distribution of posteriors must integrate to the prior.

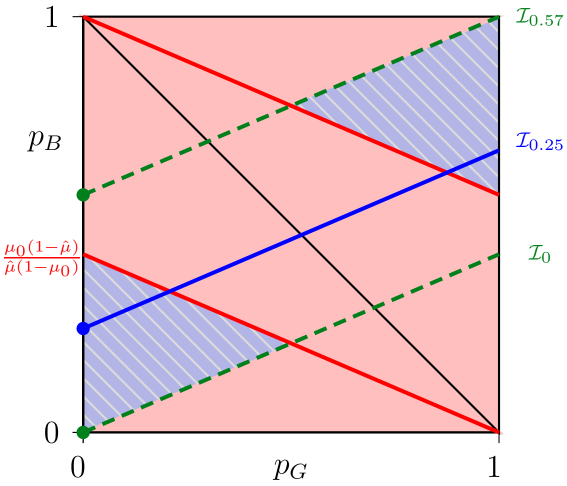


Figure 12: *The blue line, and the green dashed lines each depict an information set of the decision maker, that is a set of falsification rates that she cannot tell apart. On each of these information sets, the dot shows the only undominated strategy* (p^α_B, p^α_G) *of the agent.*

When falsification is costless, the agent is thus indifferent between any two falsification strategies in the same information set. However, when there is even mild falsification costs which increase with the levels of falsification, this indifference breaks down. We discuss this case first.

Whenever falsification is costly, as in Section 8, with a cost function $C(p_B, p_G) \geq 0$ that is increasing, any strategy $(p_B, p_G) \in \mathcal{I}_\alpha$ that does not minimize p_G (and p_B) is strictly dominated by the one that minimizes falsification rates, and thus associated costs,

$$(p^\alpha_B, p^\alpha_G) = \min\ \mathcal{I}_\alpha.$$

The cost-minimizing falsification strategies $\left\{(p^\alpha_B, p^\alpha_G)\right\}_{\alpha\in[-1,1]}$ all satisfy $p^\alpha_B + p^\alpha_G \leq 1$. Furthermore, they contain all falsification strategies of the form $(p_B, 0)$ with $p_B \leq \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}$, that is all the falsification choices that were potentially optimal in our former analysis (see Proposition 4).

Falsification strategies that do not belong to $\left\{(p^\alpha_B, p^\alpha_G)\right\}_{\alpha\in[-1,1]}$ are strictly dominated and cannot be equilibrium strategies. Therefore, when reaching information set $\mathcal{I}_\alpha$, the decision maker's equilibrium belief must be, accurately, that the agent played (p^α_B, p^α_G). Hence, our analysis of costly falsification (Section 8 and Appendix C) carries on to the case where Assumption 1 and Assumption 2 are relaxed, and all results hold. In particular, the problem of finding an optimal test can be reduced to maximizing $\mathcal{H}(\hat{\mu})$ over test functions $\mathcal{H} \in \Delta^B$ under the constraint (IC_0^c).

To extend our results in the costless case, we can follow two routes. The first option is a selection argument which consists of looking at the limit of the costly falsification problem with a vanishing cost. Consider the (linear) cost function $\varepsilon C_\lambda(p_B, p_G)$, where $C_\lambda(p_B, 0) = \lambda p_B$. Then, the following result is immediate:

Proposition 11. *The test $\mathcal{H}^*_{\varepsilon\lambda}$ is optimal under the cost function $\varepsilon C_\lambda(p_B, p_G)$, and it uniformly converges to $\mathcal{H}^*$ as $\varepsilon \to 0$.*

The second option, is to consider an agent with lexicographic preferences with approval probability as the first dimension, and an increasing falsification cost as the second dimension. Such preferences naturally capture a distaste for falsification at a given payoff level. The strict domination argument we made is still valid with these lexicographic preferences, and therefore the rest of the analysis follows as well, leading to the following result:

Theorem 3. *Under lexicographic preferences with any increasing cost function, the test $\mathcal{H}^*$ is optimal for the principal.*

10 Discussion, Robustness and Extensions

The optimal test we derive performs well despite the lack of explicit punishments or unrealistic commitment on the side of the decision maker(s). One may wonder though, whether or not a simple tool such as an admission quota would be simpler and a more compelling way to tackle cheating. Another potential concern is the extend to which out results are robust to the case where the principal does not know the prior.

Quotas We now clarify why quotas are not a good solution. Suppose the principal imposes a quota whereby no more than μ_0 fraction of items are accepted, and leaves it to the agent to point to which items should be approved. In describing this as a game, we cannot rule out that the agent would point to more than μ_0 items should be approved. A first weakness is if the agent decides which items to point to at the ex ante stage (before observing the type of his items). Then he will present a random sample of μ_0 items, implying that the DM accepts a population of items that contains a fraction $(1-\mu_0)$ of bad ones, yielding a negative payoff. If the agent picks items at the interim stage, there exists an equilibrium in which the agent indeed reports the μ_0 fraction of his items that are good. However there are other equilibria as well in which the μ_0 selected items could be anything. In particular, the only good equilibrium is not robust to small changes in the preferences of the agent. For example, if the agent favors bad items, the agent would propose those using up the quota in a way that is detrimental for the decision maker. A quota can work well if the agent slightly prefers good items to be approved or has a (possibly lexicographic) cost of pointing to bad items. But even then, if the agent presents $\mu_0+\delta$ items, the decision maker must reject items that she believes to be good with probability $\mu > \hat{\mu}$ which is inconsistent with subgame perfection. Finally, the quota is more problematic when there are multiple decision-makers since as then implementation requires coordination across them.

Uncertainty about μ_0. There are several ways to think about such uncertainty. The most natural one is that the principal and decision maker are uncertain about the fraction μ_0 of good items, while the agent knows the true μ_0. This must be the case if the agent is choosing her strategy at the interim stage. Then using our optimal test for a particular value μ_0' would lead each agent with a different realization μ_0 to falsify so as to generate the same grade distribution as an agent with μ_0' and no falsification. So an agent with $\mu_0 > \mu_0'$ would set $p_G > 0$, and an agent with $\mu_0 < \mu_0'$ would set $p_B > 0$. This implies that using such a test with a value μ_0' in the support of possible μ_0 would lead to small variations in performance when the support is sufficiently narrow. However, deriving the optimal test would require a different analysis. Another possibility would be for the principal to design menus of tests leading different types μ_0 to self select in the spirit of Kolotilin, Li, Mylovanov, and Zapechelnyuk (2016). Such an analysis and whether menus could be useful is beyond the scope of this paper.

Moral hazard and endogenous μ_0. Suppose, now, that μ_0 is endogenous in the sense that the fraction of good items in the market depends on how much effort the agent exerts. If production costs are sufficiently low, then the agent will set $\mu_0 \geq \hat{\mu}$ as, with such a prior, all items are approved regardless of the test, since any test can be turned to a completely uninformative one. If it is sufficiently costly to increase μ_0, then, in equilibrium, regardless of the test, only the least costly prior–say μ_L– is chosen. Otherwise, μ_L-agent can mimic the empirical distribution of grades of $\mu_H \neq \mu_L$ by falsifying as described in the previous paragraph. Hence the optimal test with moral hazard is our optimal test calibrated to $\mu_0 = \mu_L$.

11 Concluding Remarks

Our results have important, yet simple insights for what regulatory bodies can do to enhance the reliability of test results when agents have access to cheating technologies. First, fully revealing tests–albeit optimal in the absence of falsification–are prone to cheating, and yield the worst possible results. More generally, our analysis of a binary state, binary action setup highlights that simple (binary) tests can be fully manipulated by agent: any binary test can be turned to deliver the agent-optimal information structure. Tests that perform well have more grades than there are actions, and must assign intermediate grades with sufficiently high probability. In fact, the simple addition of a third signal can go a long way towards full optimality. We show that the optimal three-signal test delivers at least around 80% of the payoff of the optimal test, and 50% of the full information payoff. This test contains a simple practical insight: introducing a "noisy" (pooling) grade that is associated with approval in the absence of falsification, can make falsification so costly that it prevents it, rendering this noisy test much better than the (manipulated) fully informative test.

To illustrate the logic of the optimal test, consider how a four-signal approximation of our optimal test could work in practice. Such a test could have grades A, B, C, D, where A, B, C all lead to approval, but are associated with decreasingly strong beliefs about type, and D is a reject signal. In the event that some cheating is observed, grades are devalued so as to counteract the benefit of cheating to the agent. For example, if the observed extent of cheating is moderate, A, B still lead to approval, but C is devalued to a reject grade. If the extent of cheating is greater, B or even A, B can be devalued to reject grades as well.

Coming to the more abstract lessons, the no-falsification principle we derive simplifies the derivation of optimal tests since we can without loss focus on ones that induce no falsification as a best response. This result echoes the revelation principle but it is more delicate; for example, it may not hold for some costly falsification technologies. Methodologically, we introduce an elegant and tractable way to use the no-falsification constraint to analytically derive an optimal test under very general conditions.

Appendix

A Proofs without Cost

Proof of Lemma 1. Let $\mathcal{H} \in \Delta^B$. By convexity, $\mathcal{H}$ has a left derivative everywhere on $(0, 1]$, let $H(\mu)$ be the left derivative of $\mathcal{H}$ at μ. Furthermore, H is piecewise-continuous, everywhere left-continuous, and weakly increasing on $(0, 1]$. Then, we can define $H(0) = \lim_{\mu\to 0} H(\mu)$. Because, $\mathcal{H}$ is increasing, H is non-negative. It is also bounded above by 1. Suppose not, so that $H(\mu) > 1$ for some μ. Because H is left-continuous, there must be an interval $[\mu - \varepsilon, \mu]$ to the left of μ such that $H(x) > 1$ for all $x \in [\mu - \varepsilon, \mu]$, so we can choose $x < 1$ such that $H(x) > 1$. By convexity, we must have $\mathcal{H}(1) - \mathcal{H}(x) \geq H(x)(1 - x) > 1 - x$. Since $\mathcal{H}(1) = 1 - \mu_0$, this implies $\mathcal{H}(x) < x - \mu_0$, but then $\mathcal{H}$ would violate the lower bound condition on Δ^B.

Next, let H be a probability measure on $[0, 1]$ with mean μ_0, and also the associated pseudo cdf. Define $\mathcal{H}(\mu) = \int_0^\mu H(x)dx$. This function is increasing since H is nonnegative. It is also convex as the integral of a non-decreasing function. The condition on the mean implies that $\mathcal{H}(1) = 1 - \mu_0$, and $\mathcal{H}(0) = 0$ by definition. Suppose that for some $x \in (0, 1)$, $\mathcal{H}(x) > x(1 - \mu_0)$. Then, convexity of $\mathcal{H}$ would imply that

$$\mathcal{H}(1) \geq \mathcal{H}(x) + \frac{\mathcal{H}(x) - \mathcal{H}(0)}{x}(1 - x) = \frac{\mathcal{H}(x)}{x} > 1 - \mu_0,$$

a contradiction. Similarly, if for some $x > \mu_0$, we had $\mathcal{H}(x) < (x - \mu_0)^+$, convexity would imply that $\mathcal{H}(0) < 0$, a contradiction. □

Proof of Lemma 3. Let $\lambda(\mu) \equiv \frac{H_G(d\mu)}{H_B(d\mu)}$ denote the likelihood ratio induced by the test when the signal realization (and the belief in the absence of falsification) is a small interval $d\mu$ centered on μ. In the presence of falsification, the signal μ observed as a result of the test can no longer be identified with the the belief formed by the principal. Specifically, by Bayes rule, the belief $\tilde{\mu}$ that is formed when signal μ is generated satisfies

$$\tilde{\mu} = \frac{\mu_0\tilde{\lambda}(\mu)}{\mu_0\tilde{\lambda}(\mu) + 1 - \mu_0}, \tag{3}$$

where

$$\tilde{\lambda}(\mu) = \frac{F_G(d\mu)}{F_B(d\mu)} = \frac{(1 - p_G)H_G(d\mu) + p_G H_B(d\mu)}{(1 - p_B)H_B(d\mu) + p_B H_G(d\mu)} = \frac{(1 - p_G)\lambda(\mu) + p_G}{p_B\lambda(\mu) + 1 - p_B}$$

is the new relevant likelihood ratio. This expression is increasing in λ over $[0, \infty)$ whenever $p_B + p_G < 1$, meaning that the post-falsification belief is increasing in the initial belief. By contrast, if $p_B + p_G < 1$, it is decreasing in λ. This relationship can be inverted to get

$$\lambda(\mu) = \frac{(1-p_B)\tilde{\lambda}(\mu) - p_G}{1 - p_G - p_B\tilde{\lambda}(\mu)}.$$

A simple rewriting of (3) also gives us: $\tilde{\lambda}(\mu) = \frac{\tilde{\mu}(1-\mu_0)}{\mu_0(1-\tilde{\mu})}$. Using these expressions, we can write the signal, and original belief, as a function of the post-falsification belief:

$$\begin{aligned}
\mu &= \frac{\mu_0}{\mu_0 + (1-\mu_0)\lambda(\mu)^{-1}} \\
&= \frac{\mu_0}{\mu_0 + (1-\mu_0)\frac{1-p_G-p_B\tilde{\lambda}(\mu)}{(1-p_B)\tilde{\lambda}(\mu)-p_G}} \\
&= \frac{\mu_0}{\mu_0 + (1-\mu_0)\frac{1-p_G-p_B\frac{1-\mu_0}{\mu_0}\frac{\tilde{\mu}}{1-\tilde{\mu}}}{(1-p_B)\frac{1-\mu_0}{\mu_0}\frac{\tilde{\mu}}{1-\tilde{\mu}}-p_G}} \\
&= \frac{\mu_0}{\mu_0 + (1-\mu_0)\frac{\mu_0(1-p_G)-\tilde{\mu}(p_B+\mu_0(1-p_G-p_B))}{\tilde{\mu}(1-p_B-\mu_0(1-p_B-p_G))-\mu_0 p_G}} \\
&= \frac{\mu_0\left(\tilde{\mu}\left(1-p_B-\mu_0(1-p_B-p_G)\right)-\mu_0 p_G\right)}{\mu_0(1-p_G)-\tilde{\mu}\left(p_B+\mu_0(1-p_G-p_B)\right)+\mu_0(\tilde{\mu}-\mu_0)} \\
&= \frac{\mu_0\tilde{\mu}\big(1-p_B-\mu_0(1-p_B-p_G)\big)-\mu_0^2 p_G}{\tilde{\mu}\big(\mu_0(p_B+p_G)-p_B\big)+\mu_0(1-\mu_0)-\mu_0 p_G} \\
&= \mu_0\frac{(1-\mu_0)\tilde{\mu}-\mu_0(1-\tilde{\mu})p_G-(1-\mu_0)\tilde{\mu}p_B}{\mu_0(1-\mu_0)-\mu_0(1-\tilde{\mu})p_G-(1-\mu_0)\tilde{\mu}p_B}.
\end{aligned}$$

It is easy to see that $\tilde{\mu}$ lies in $\left[\frac{\mu_0 p_G}{\mu_0 p_G+(1-\mu_0)(1-p_B)}, \frac{\mu_0(1-p_G)}{\mu_0(1-p_G)+(1-\mu_0)p_B}\right]$. The remaining points follow from easy calculations. □

Proof of Lemma 2. We show the proof for F_G, it is similar for F_B. Consider the joint probability that a certain item is of the good type, and the information structure generates a belief in $[0, \mu)$ for this item. This probability can be written as $\mu_0 F_G(\mu)$, or as $\int_0^\mu xF(dx)$. By integration by parts, the latter is equal to $\mu F(\mu) - \mathcal{F}(\mu)$, which concludes the proof. □

Proof of Proposition 3. If $p_B + p_G = 1$, the resulting information structure is uninformative, the principal has belief μ_0 regardless of the signal and does not approve. Next, we treat the case $p_B + p_G < 1$. Because μ_0 is the prior, it must lie in the interval $\left[\underline{\mu}, \overline{\mu}\right]$. $\hat{\mu}$, however, need not lie in this interval, and, if it does not, the principal never approves. This is the case if the upper bound of the interval is below $\hat{\mu}$, that is

$$\frac{\mu_0(1-p_G)}{\mu_0(1-p_G)+(1-\mu_0)p_B} < \hat{\mu} \quad \Leftrightarrow \quad p_B > \frac{\mu_0(1-\hat{\mu})}{(1-\mu_0)\hat{\mu}}(1-p_G).$$

When this is not the case, the principal approves for beliefs above $\hat{\mu}$, that is for signals above

$$\begin{aligned}\hat{\mu}(p_B,p_G) &= \frac{\mu_0\hat{\mu}\big(1-p_B-\mu_0(1-p_B-p_G)\big)-\mu_0^2 p_G}{\hat{\mu}\big(\mu_0(p_B+p_G)-p_B\big)+\mu_0(1-\mu_0)-\mu_0 p_G} \\ &= \mu_0\frac{(1-\mu_0)\hat{\mu}-\mu_0(1-\hat{\mu})p_G-(1-\mu_0)\hat{\mu}p_B}{\mu_0(1-\mu_0)-\mu_0(1-\hat{\mu})p_G-(1-\mu_0)\hat{\mu}p_B}.\end{aligned}$$

A simple calculation shows that this $\hat{\mu}(p_B,p_G)$ increases with p_B and p_G for $p_B+p_G<1$.

Finally, consider the case $p_B+p_G>1$. Then, the belief transformation is decreasing, and the decision maker will therefore approve when signals are below $\hat{\mu}(p_B,p_G)$. As previously, $\hat{\mu}$ may not lie in the interval $\big[\underline{\mu},\overline{\mu}\big]$. Now, it is the case if $\hat{\mu}$ lies below $\underline{\mu}$, that is

$$\frac{\mu_0 p_G}{\mu_0 p_G+(1-\mu_0)(1-p_B)} > \hat{\mu} \quad \Leftrightarrow \quad p_B > 1-\frac{\mu_0(1-\hat{\mu})}{(1-\mu_0)\hat{\mu}}p_G.$$

A simple calculation shows that $\hat{\mu}(p_B,p_G)$ decreases with p_B and p_G for $p_B+p_G>1$. □

To prove Proposition 4 we need the help of the following lemma.

Lemma 5. *For every $\mu\in[\mu_0,1]$, $\mathcal{H}(\mu)-(\mu-\mu_0)H(\mu)\geq 0$, and the inequality is strict if and only if $H(\mu)<1$. Furthermore, this expression is nonincreasing in μ.*

Proof. Since $H(\mu)\leq 1$, we have $\mathcal{H}(\mu)-(\mu-\mu_0)H(\mu)\geq\mathcal{H}(\mu)-(\mu-\mu_0)\geq 0$ by definition of Δ^B, since $(\mu-\mu_0)^+$ is the lower bound of Δ^B. The first inequality is strict if $H(\mu)<1$. Then, note that, for any $\mu>\mu'>\hat{\mu}$, we have, by convexity

$$\begin{aligned}\mathcal{H}(\mu)-(\mu-\mu_0)H(\mu)-\mathcal{H}(\mu')+(\mu'-\mu_0)H(\mu') &\leq H(\mu')(\mu-\mu')-(\mu-\mu_0)H(\mu)+(\mu'-\mu_0)H(\mu') \\ &\leq \big(H(\mu')-H(\mu)\big)(\mu-\mu_0)\leq 0\end{aligned}$$

□

Proof of Proposition 4. If $H(\hat{\mu})=1$, then $H\big(\hat{\mu}(p_B,p_G)\big)=1$ for any falsification strategy. Therefore, the first term in the expression of $\Pi(p_B,p_G)$ is null, and, by Lemma 5, so is the second term. Hence the payoff of the agent is null, regardless of her falsification strategy. Furthermore, the decision maker approves with probability 0, and therefore her payoff is null.

If $H(\hat{\mu}) < 1$, then no falsification gives the agent a strictly positive payoff. Therefore any optimal falsification must be such that $p_B \leq \frac{\mu_0(1-\hat{\mu})}{\hat{\mu}(1-\mu_0)}(1-p_G)$, that is, it must lie below the red line in Figure 5. In addition, it must satisfy $H\big(\hat{\mu}(p_B, p_G)\big) < 1$ and $p_B \geq \frac{\mu_0}{1-\mu_0}p_G$. The second inequality corresponds to the region above the dashed green line in Figure 5. Indeed, a falsification strategy such that $H\big(\hat{\mu}(p_B, p_G)\big) = 1$ would yield a null payoff, and we know that the agent can do better. Then at any potentially optimal falsification strategy, we have $\mathcal{H}\big(\hat{\mu}(p_B, p_G)\big) - \big(\hat{\mu}(p_B, p_G) - \mu_0\big)H\big(\hat{\mu}(p_B, p_G)\big) > 0$ by Lemma 5. Suppose that $p_B < \frac{\mu_0}{1-\mu_0}p_G$. Then we would have

$$\Pi(p_B, p_G) < 1 - H\big(\hat{\mu}(p_B, p_G)\big) \leq 1 - H(\hat{\mu}),$$

so the agent would be better off by not falsifying.

Next, let (p_B, p_G) be a falsification strategy that satisfies all these criteria, so that it is potentially optimal. Then $\Pi(p_B, p_G)$ is decreasing in p_G. Indeed, the first term, $1 - H\big(\hat{\mu}(p_B, p_G)\big)$, is nonincreasing in p_G since $\hat{\mu}(p_B, p_G)$ is nondecreasing in p_G. Then $\mathcal{H}\big(\hat{\mu}(p_B, p_G)\big) - \big(\hat{\mu}(p_B, p_G) - \mu_0\big)H\big(\hat{\mu}(p_B, p_G)\big) > 0$ is nonincreasing in p_G by Lemma 5, and $\frac{p_B}{\mu_0} - \frac{p_G}{1-\mu_0} > 0$ is decreasing in p_G.

□

Proof of Proposition 6. We have already proved optimality, so the only thing that remains to be proved is that this experiment indeed corresponds to the one we identified in Proposition 2, that is they generate the same belief distributions. The first experiment generates probability $(1-\mu_0)(1-\pi_B^*)$ on 0, and the following calculation shows that this is equal to $H(0) = \kappa$,

$$\begin{aligned}(1-\mu_0)(1-\pi_B^*) &= 1 - \mu_0 - \frac{\mu_0(1-\hat{\mu})^2}{\hat{\mu}(2-\hat{\mu})} \\ &= \frac{(1-\mu_0) - (1-\hat{\mu})^2}{\hat{\mu}(2-\hat{\mu})},\end{aligned}$$

which concludes the proof since other probabilities must coincide as well for both experiments to generate an average belief of μ_0 and have the same atoms. □

Proof of Theorem 1. Here, we prove the missing steps in the proof of the theorem.

Step 1. The first step is to prove that $\mathcal{H}^*$ is indeed in Δ^B. Note that $\mathcal{H}^*$ is continuously differentiable, and to show that it is in Δ^B, it is sufficient to show that its derivative H^* is indeed a pseudo cdf. Hence, we show that H^* is nondecreasing and bounded between 0 and 1. First, note that κ^* is positive. Therefore $H^*(\mu)$ is positive for $\mu \leq \hat{\mu}$. For $\mu > \hat{\mu}$, we know that

$$H^*(\mu) = \frac{\kappa^* \hat{\mu}}{\mu} + \frac{\mu - \hat{\mu}}{\mu(\mu - \mu_0)} \mathcal{H}^*(\mu),$$

and, since $\mathcal{H}^*(\mu)$ is clearly positive, so is $H^*(\mu)$.

Next, we show that H^* is non-decreasing. This is immediate on $\big[0, \hat{\mu}\big]$. For $\mu \geq \hat{\mu}$, we start by calculating the integral in the expression of $\psi(\mu)$

$$\begin{aligned}\log\big(\psi(\mu)\big) = \int_{\hat{\mu}}^{\mu} \frac{x - \hat{\mu}}{x(x - \mu_0)} dx &= \int_{\hat{\mu}}^{\mu} \frac{1}{x - \mu_0} dx - \int_{\hat{\mu}}^{\mu} \frac{\hat{\mu}}{x(x - \mu_0)} dx \\ &= \Big[\log(x - \mu_0)\Big]_{\hat{\mu}}^{\mu} - \frac{\hat{\mu}}{\mu_0}\Big[2\log(x - \mu_0) - \log\big(x(x - \mu_0)\big)\Big]_{\hat{\mu}}^{\mu} \\ &= \log\left(\frac{\mu - \mu_0}{\hat{\mu} - \mu_0}\right) + \frac{\hat{\mu}}{\mu_0} \log\left(\frac{\mu(\hat{\mu} - \mu_0)}{\hat{\mu}(\mu - \mu_0)}\right).\end{aligned}$$

Replacing in the expression of $\mathcal{H}^*(\mu)$, we get

$$\mathcal{H}^*(\mu) = \kappa^* \hat{\mu}(\mu - \mu_0) \left(\frac{\mu}{\mu - \mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}} \left\{(\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} \hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}} + \int_{\hat{\mu}}^{\mu} (x - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} x^{-\frac{\hat{\mu}}{\mu_0} - 1} dx\right\}.$$

The remaining integral is

$$\int_{\hat{\mu}}^{\mu} (x - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} x^{-\frac{\hat{\mu}}{\mu_0} - 1} dx = \left[\frac{1}{\hat{\mu}} \left(\frac{x - \mu_0}{x}\right)^{\frac{\hat{\mu}}{\mu_0}}\right]_{\hat{\mu}}^{\mu} = \frac{1}{\hat{\mu}} \left(\frac{\mu - \mu_0}{\mu}\right)^{\frac{\hat{\mu}}{\mu_0}} - \frac{1}{\hat{\mu}} \left(\frac{\hat{\mu} - \mu_0}{\hat{\mu}}\right)^{\frac{\hat{\mu}}{\mu_0}}.$$

Finally, we obtain

$$\mathcal{H}^*(\mu) = \kappa^*(\mu - \mu_0) \left\{1 + \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} \hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}} \left(\frac{\mu}{\mu - \mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}}\right\}. \tag{4}$$

Differentiating, we find

$$H^*(\mu) = \kappa^* \left\{1 + \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0} - 1} \hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}} (\mu - \hat{\mu}) (\mu - \mu_0)^{-\frac{\hat{\mu}}{\mu_0}} \mu^{\frac{\hat{\mu}}{\mu_0} - 1}\right\}.$$

Hence H^* is continuously differentiable on $[\hat{\mu}, 1]$. We denote its derivative by h^*. Differentiating again, we get

$$h^*(\mu) = \kappa^* \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0}} \hat{\mu}^{1-\frac{\hat{\mu}}{\mu_0}} (\mu - \mu_0)^{-\frac{\hat{\mu}}{\mu_0}-1} \mu^{\frac{\hat{\mu}}{\mu_0}-2}. \tag{5}$$

Hence $h^*(\mu)$ is strictly positive on $\left[\hat{\mu}, 1\right]$, and H^* is strictly increasing.

To conclude step 1, we only need to show that $H^*(1) \leq 1$. By (IDE), we have $H^*(1) = \kappa^* \hat{\mu} + 1 - \hat{\mu}$. Hence, we need to show $\kappa^* \leq 1$. Using (4) and the condition $\mathcal{H}^*(1) = 1 - \mu_0$, we have

$$1 - \mu_0 = \mathcal{H}(1) = \kappa^*(1 - \mu_0) \underbrace{\left\{ 1 + \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0}-1} \hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}-2} \left(\frac{1}{1 - \mu_0} \right)^{\frac{\hat{\mu}}{\mu_0}} \right\}}_{\geq 1},$$

which concludes the proof.

Step 3. Suppose that $\mathcal{H}$ is an optimal experiment, that is not less informative than $\mathcal{H}^*$. By Lemma 4, we can as well take $\mathcal{H}$ to be linear since the linear transformation invoked in this lemma is above the original experiment, and therefore more informative. Since $\mathcal{H}$ is optimal, we must have $\mathcal{H}(\mu) = \mathcal{H}^*(\mu) = \kappa^* \mu$, for all $\mu \leq \hat{\mu}$. For $\mathcal{H}$ not to be less informative than $\mathcal{H}^*$, there must therefore exist some $\mu \in \left(\hat{\mu}, 1\right)$ such that $\mathcal{H}(\mu) > \mathcal{H}^*(\mu)$. Since $\mathcal{H} - \mathcal{H}^*$ is continuous and $\mathcal{H}(1) = \mathcal{H}^*(1)$, we can find the lowest point x above μ at which $\mathcal{H}(x) = \mathcal{H}^*(x)$. Let $\tilde{\mu}$ be this point. Then $\mathcal{H}(x) > \mathcal{H}^*(x)$ for every $x \in \left[\mu, \tilde{\mu}\right)$. But then, there must exist a subset X of $[\mu, \tilde{\mu}]$ with positive measure, such that $H(x) < H^*(x)$ for every $x \in X$, as otherwise, we would have $\mathcal{H}(\tilde{\mu}) - \mathcal{H}(\mu) = \int_{\mu}^{\tilde{\mu}} H(\mu) d\mu \geq \int_{\mu}^{\tilde{\mu}} H^*(\mu) d\mu = \mathcal{H}^*(\tilde{\mu}) - \mathcal{H}^*(\mu)$, a contradiction. Then take $x \in X$. We have $H(x) < H^*(x)$ and $\mathcal{H}(x) > \mathcal{H}^*(x)$. Therefore

$$xH(x) - \frac{x - \hat{\mu}}{x - \mu_0} < xH^*(x) - \frac{x - \hat{\mu}}{x - \mu_0} \mathcal{H}^*(x) = \kappa^* \hat{\mu},$$

and $\mathcal{H}$ must violate (IC_0').

□

Proof of Proposition 7. We have already proved that $\mathcal{H}^*$ is continuously differentiable and admits a density on $\big[\hat{\mu}, 1\big)$, which is given by (5). Differentiating (5), we get

$$h^{*\prime}(\mu) = -\kappa^* \mu_0 (\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0}} \hat{\mu}^{1-\frac{\hat{\mu}}{\mu_0}} (\mu - \mu_0)^{-\frac{\hat{\mu}}{\mu_0}-2} \mu^{\frac{\hat{\mu}}{\mu_0}-3} \big(\mu + \hat{\mu} + 2(\mu - \mu_0)\big) < 0.$$

Note that we can also write

$$h^{*\prime}(\mu) = \frac{h(\mu)}{\mu(\mu - \mu_0)} \Big\{ -\hat{\mu} - \mu - 2(\mu - \mu_0) \Big\}.$$

Differentiating the expressions in Lemma 2, we obtain that the densities of the belief distributions generated by the two types on $\big[\hat{\mu}, 1\big)$ are

$$h^*_G(\mu) = \frac{\mu}{\mu_0} h^*(\mu),$$

and

$$h^*_B(\mu) = \frac{1 - \mu}{1 - \mu_0} h(\mu).$$

A quick calculation yields

$$h^{*\prime}_G(\mu) = \frac{h^*(\mu)}{\mu_0(\mu - \mu_0)} \Big\{ -\hat{\mu} - \mu - (\mu - \mu_0) \Big\} < 0,$$

and

$$h^{*\prime}_B(\mu) = \frac{h^*(\mu)}{(1 - \mu_0)(\mu - \mu_0)\mu} \Big\{ -(1 - \mu)\Big[\hat{\mu} + \mu + (\mu - \mu_0)\Big] - \mu(\mu - \mu_0) \Big\} < 0.$$

To prove first-order stochastic dominance, we can use the expressions in Lemma 2 to get

$$H^*_G(\mu) - H^*_B(\mu) = \frac{1}{\mu_0(1 - \mu_0)} \Big\{ (\mu - \mu_0) H^*(\mu) - \mathcal{H}^*(\mu) \Big\}.$$

We know by Lemma 5 that this expression is negative for $\mu \geq \mu_0$. For $\mu < \mu_0$, we have $H^*(\mu) = \kappa^*$, and $\mathcal{H}^*(\mu) = \kappa^* \mu$, therefore

$$H^*_G(\mu) - H^*_B(\mu) = -\frac{\kappa^*}{1 - \mu_0} < 0.$$

□

Proof of Proposition 8. Pareto efficiency can be seen graphically. Fixing a payoff for the principal, that is a value of $\mathcal{F}(\hat{\mu})$, the information structure that maximizes the payoff of the agent is the one that minimizes the left derivative $F(\hat{\mu})$, while keeping the function $\mathcal{F}$ convex, and under the constraint that $\mathcal{F}(0,0)$. The only possibility is therefore to make $\mathcal{F}$ linear between $(0,0)$, and $(\hat{\mu}, \mathcal{F}(\hat{\mu}))$.

For the performance ratio, consider first $\mathcal{H}^*_{3S}$. Recalling that the payoff of the principal is equal to $\mu_0 - \hat{\mu} + \mathcal{F}(\hat{\mu})$, the performance ratio is

$$\frac{\mu_0 - \hat{\mu} + \kappa^*_{3S}\hat{\mu}}{\mu_0(1-\hat{\mu})} = \frac{1}{2-\hat{\mu}}.$$

Interestingly, this ratio is independent of μ_0. It is easy to see that it is bounded below by 1/2, and that this bound is strict.

Next, the performance ratio of $\mathcal{H}^*$ must by construction be greater than the performance ratio of $\mathcal{H}^*_{3S}$, and hence above 1/2. To show that this bound is strict, we construct a sequence of pairs $(\mu_0, \hat{\mu})$ such that the corresponding performance ratio approaches 1/2. The performance ratio of $\mathcal{H}^*$ is given by

$$\begin{aligned} R(\mu_0, \hat{\mu}) = \frac{\mu_0 - \hat{\mu} + \kappa^*_{3S}\hat{\mu}}{\mu_0(1-\hat{\mu})} &= \frac{\mu_0 - \hat{\mu} + \hat{\mu}\left(1 + \frac{\mu_0}{\hat{\mu}-\mu_0}\left(\frac{\hat{\mu}-\mu_0}{\hat{\mu}(1-\mu_0)}\right)^{\frac{\hat{\mu}}{\mu_0}}\right)^{-1}}{\mu_0(1-\hat{\mu})} \\ &= \frac{1 - \left(\frac{\hat{\mu}-\mu_0}{\hat{\mu}(1-\mu_0)}\right)^{\frac{\hat{\mu}}{\mu_0}}}{(1-\hat{\mu})\left(1 + \frac{\mu_0}{\hat{\mu}-\mu_0}\left(\frac{\hat{\mu}-\mu_0}{\hat{\mu}(1-\mu_0)}\right)^{\frac{\hat{\mu}}{\mu_0}}\right)}. \end{aligned}$$

The sequence we consider is defined for $n \geq 2$ by

$$\begin{aligned} \mu_0^n &= \frac{1}{n}, \\ \hat{\mu}^n &= \frac{1}{n} + \frac{1}{n^2}. \end{aligned}$$

Hence

$$R(\mu_0^n, \hat{\mu}^n) = \frac{1}{1-\hat{\mu}^n} \frac{1 - \left(\frac{n}{(n-1)(n+1)}\right)^{1+\frac{1}{n}}}{1 + n\left(\frac{n}{(n-1)(n+1)}\right)^{1+\frac{1}{n}}}$$

As $n \to \infty$, the term $\frac{1}{1-\hat{\mu}^n}$ converges to 1, and the term $\left(\frac{n}{(n-1)(n+1)}\right)^{1+\frac{1}{n}}$ converges to 0. For the remaining term, we can write:

$$n\left(\frac{n}{(n-1)(n+1)}\right)^{1+\frac{1}{n}} = \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{1}{1-\frac{1}{n}}\right)^{1+\frac{1}{n}}\left(\frac{1}{1+n}\right)^{\frac{1}{n}}.$$

Since each of the terms in this product converges to 1 as $n \to \infty$, we have

$$\lim_{n\to\infty} R\big(\mu_0^n, \hat{\mu}^n\big) = \frac{1}{2}.$$

□

Proof of Theorem 2. We proceed in three steps.

Step 1: Optimality: Optimality works as in the proof of Theorem 1.

Step 2: $\mathcal{H}_{FI}(\mu) > \mathcal{H}^*_\lambda(\mu) > \mathcal{H}^*(\mu)$. Using the expressions of $\mathcal{H}^*$ and $\mathcal{H}^*_\lambda$, we can write the difference of the two functions for each $\mu \geq \hat{\mu}$ as

$$\mathcal{H}^*_\lambda(\mu) = \mathcal{H}^*(\mu) + \lambda\mu_0\psi(\mu)\frac{G(\mu)}{G(1)}(B(1) - B(\mu)) \tag{6}$$

where $B(y) \equiv \int_{\hat{\mu}}^{y} \frac{x-\hat{\mu}}{x(x-\mu_0)\psi(x)}dx$ and $G(y) \equiv \left(1 + \int_{\hat{\mu}}^{y} \frac{1}{x\psi(x)}dx\right)$ which, because $B(1) - B(\mu) > 0$ and all other terms are positive, implies that $\mathcal{H}_{FI}(\mu) > \mathcal{H}^*_\lambda(\mu)$ on $\big(0, 1\big)$.

To see how we can get (6), note that

$$\kappa^* = \frac{1-\mu_0}{\hat{\mu}\psi(1)\underbrace{\left(1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx\right)}_{G(1)}} = \frac{1-\mu_0}{\hat{\mu}\psi(1)G(1)} \tag{7}$$

which implies the following expression for $\mathcal{H}^*(\mu)$:

$$\mathcal{H}^*(\mu) = \kappa^*\hat{\mu}\psi(\mu)\left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx\right) = \kappa^*\hat{\mu}\psi(\mu)G(\mu) = (1-\mu_0)\frac{\psi(\mu)G(\mu)}{\psi(1)G(1)}. \tag{8}$$

Note also that

$$\kappa^*_\lambda = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \lambda\frac{\mu_0}{\hat{\mu}}\underbrace{\int_{\hat{\mu}}^1 \frac{x-\hat{\mu}}{x(x-\mu_0)\psi(x)}dx}_{B(1)}\right)\left(1+\int_{\hat{\mu}}^1 \frac{1}{x\psi(x)}dx\right)^{-1} = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \lambda\frac{\mu_0}{\hat{\mu}}B(1)\right)G(1)^{-1},$$

or, combined with (7):

$$\kappa^*_\lambda = \kappa^* + \lambda\frac{\mu_0}{\hat{\mu}}\frac{B(1)}{G(1)},$$

which allows us to write:

$$\mathcal{H}^*_\lambda(\mu) = \hat{\mu}\psi(\mu)\left[\kappa^*_\lambda G(\mu) - \lambda\frac{\mu_0}{\hat{\mu}}B(\mu)\right]$$

which gives us (6). Now replacing (8) to (6) we obtain:

$$\mathcal{H}^*_\lambda(\mu) = (1-\mu_0)\frac{\psi(\mu)G(\mu)}{\psi(1)G(1)} + \lambda\mu_0\psi(\mu)\frac{G(\mu)}{G(1)}(B(1)-B(\mu)). \tag{9}$$

Finally noting that $\mathcal{H}_{FI}(\mu)$ is a solution to the differential equation when $\lambda = 1-\mu_0$, (9) implies that $\mathcal{H}_{FI}(\mu) > \mathcal{H}^*_\lambda(\mu)$ when $\lambda < 1-\mu_0$.

Step 3: $\mathcal{H}^*_\lambda \in \Delta^B$: After some algebra, we get the following expression of $\mathcal{H}^*_\lambda$ to the right of $\hat{\mu}$.

$$\mathcal{H}^*_\lambda(\mu) = \kappa^*_\lambda\mu + (\kappa^*_\lambda - \lambda)\mu_0\left\{\left(\frac{\mu}{\hat{\mu}}\right)^{\frac{\hat{\mu}}{\mu_0}}\left(\frac{\hat{\mu}-\mu_0}{\mu-\mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}-1} - 1\right\}.$$

This implies

$$\kappa^*_\lambda = \frac{1-\mu_0+\lambda\mu_0\left[\hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}}\left(\frac{\hat{\mu}-\mu_0}{1-\mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}-1} - 1\right]}{1-\mu_0+\mu_0\hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}}\left(\frac{\hat{\mu}-\mu_0}{1-\mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}-1}} > \lambda.$$

Differentiating, we get

$$H^*_\lambda(\mu) = \kappa^*_\lambda + (\kappa^*_\lambda - \lambda)\mu_0\hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}}(\hat{\mu}-\mu_0)^{\frac{\hat{\mu}}{\mu_0}-1}(\mu-\hat{\mu})\mu^{\frac{\hat{\mu}}{\mu_0}-1}(\mu-\mu_0)^{-\frac{\hat{\mu}}{\mu_0}} \geq \kappa^*_\lambda.$$

And differentiating again

$$h_{\lambda}^{*}(\mu) = (\kappa_{\lambda}^{*} - \lambda)\mu_0\hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}}(\hat{\mu} - \mu_0)^{\frac{\hat{\mu}}{\mu_0}}\mu^{\frac{\hat{\mu}}{\mu_0}-2}(\mu - \mu_0)^{-\frac{\hat{\mu}}{\mu_0}-1} > 0. \tag{10}$$

Hence, we have convexity. Combined with $\mathcal{H}_{FI}(\mu) > \mathcal{H}_{\lambda}^{*}(\mu) > \mathcal{H}^{*}(\mu)$, this proves that $\mathcal{H}_{\lambda}^{*}$ is in Δ^{B}.

□

Proof of Proposition 9. The proof can be obtained from (10), by proceeding as in the proof of Proposition 7. □

Proof of Proposition 10. Take $1 - \mu_0 \geq \lambda' > \lambda \geq 0$. Then we can prove $\mathcal{H}_{\lambda'}^{*}(\mu) > \mathcal{H}_{\lambda}^{*}(\mu)$ exactly in the same way as we prove $\mathcal{H}_{FI}(\mu) > \mathcal{H}_{\lambda}^{*}(\mu) > \mathcal{H}^{*}(\mu)$ in the proof of Theorem 2.

□

B Observability and No Limits to Falsification Rates

In this Appendix we explain why removing falsification limits while assuming perfect observability leads to manipulations. Under $\mathcal{H}^{*}$, choosing $p_B + p_G > 1$ leads the decision maker to form beliefs below $\hat{\mu}$ whenever she observes a signal above $\hat{\mu}$. So all signals that would have led to approval under no falsification now lead to rejection. However, the reject signal 0 may now lead to a belief above $\hat{\mu}$. In fact, the optimal falsification rates with $p_B + p_G > 1$ must lead the decision maker to form belief $\hat{\mu}$ when she sees signal 0. This optimal falsification strategy is described in the following proposition, and illustrated in Figure 13.

Proposition 12. *Under Assumption 1, but without limits on falsification rates, the optimal falsification strategy under $\mathcal{H}^{*}$ is to choose $p_G = 1$, and $p_B = \frac{\hat{\mu} - \mu_0}{\hat{\mu}(1-\mu_0)}$. The agent gets a payoff of $\mu_0/\hat{\mu}$, whereas the principal and decision maker get a null payoff.*

Proof. Optimality of the proposed falsification strategy among those such that $p_B + p_G > 1$ follows from the arguments just given. Among other falsification strategies, we know that $(0, 0)$ is optimal, by design of $\mathcal{H}^{*}$. To show that the proposed falsification strategy is optimal among all available ones, we just need to show that the payoff it yields for the agent, $\mu_0/\hat{\mu}$ is greater

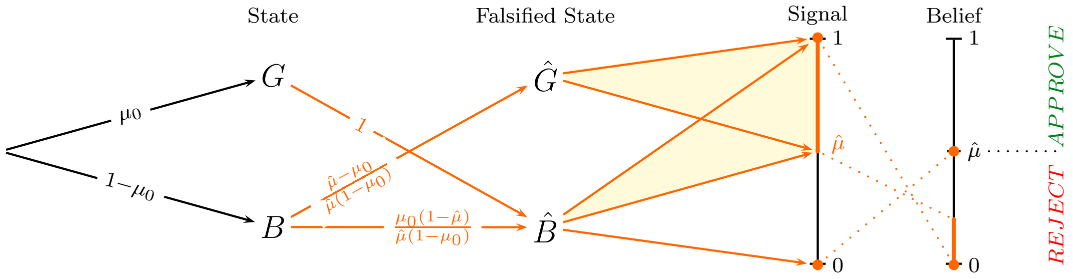


Figure 13: *Manipulating $\mathcal{H}^*$, under perfect observability and no limits on falsification.*

than the payoff the agent gets under $(0,0)$. The latter is given by $1 - H^*(\hat\mu) = 1 - \kappa^*$. Hence we need to show that

$$\kappa^* = \frac{1}{1+\mu_0(\hat\mu-\mu_0)^{\frac{\hat\mu}{\mu_0}-1}\hat\mu^{-\frac{\hat\mu}{\mu_0}}(1-\mu_0)^{-\frac{\hat\mu}{\mu_0}}} > \frac{\hat\mu-\mu_0}{\hat\mu},$$

or, after simplification,

$$1 < \left(\frac{\hat\mu-\mu_0}{\hat\mu(1-\mu_0)}\right)^{\frac{\hat\mu}{\mu_0}},$$

which holds as $\hat\mu > \mu_0$. □

Thus, under perfect observability, the agent can profitably deviate from no-falsification to falsification rates such that $p_B + p_G > 1$ when the principal uses test $\mathcal{H}^*$. But this problem vanishes if we also relax the perfect observability assumption Assumption 1, and instead allow the decision maker to learn about cheating only through the cross-sectional distribution of test results as we do in Section 9.

References

Bizzotto, J., J. Rudiger, and A. Vigier (2016): "Delegated Certification," Working paper.

Blackwell, D. (1951): "The Comparison of Experiments," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, ed. by J. Neyman, University of California Press, Berkeley, 93–102.

——— (1953): "Equivalent Comparisons of Experiments," *Annals of Mathematical Statistics*, 24, 265–272.

Boleslavsky, R. and K. Kim (2017): "Bayesian Persuasion and Moral Hazard," .

Chassang, S. and J. Ortner (2016): "Making Corruption Harder: Asymmetric Information, Collusion and Crime," Working paper.

Cohn, J. B., U. Rajan, and G. Strobl (2016): "Credit ratings: strategic issuer disclosure and optimal screening," Working paper.

Condorelli, D. and B. Szentes (2016): "Buyer-Optimal Demand and Monopoly Pricing," Tech. rep., Mimeo, London School of Economics and University of Essex.

Cunningham, T. and I. Moreno de Barreda (2015): "Equilibrium Persuasion," Working paper.

Gentzkow, M. and E. Kamenica (2014): "Costly Persuasion," *American Economic Review*, 104, 457–462.

——— (2016a): "Bayesian persuasion with multiple senders and a rich signal space," .

——— (2016b): "A Rotschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review: Papers and Proceedings*, 106, 597–601.

Golosov, M. and A. Tsyvinski (2007): "Optimal Taxation with Endogenous Insurance Markets," *Quarterly Journal of Economics*, 122, 487–534.

Grochulski, B. (2007): "Optimal Nonlinear Income Taxation with Costly Tax Avoidance," *Economic Quarterly - Richmond Fed.*

Hörner, J. and N. S. Lambert (2016): "Motivational ratings," Working paper.

Jackson, M. O. and H. F. Sonnenschein (2007): "Overcoming incentive constraints by linking decisions," *Econometrica*, 75, 241–257.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Kolotilin, A. (2016): "Optimal Information Disclosure: A Linear Programming Approach," Working paper.

Kolotilin, A., M. Li, T. Mylovanov, and A. Zapechelnyuk (2016): "Persuasion of a Privately Informed Receiver," Working paper.

Lacker, J. M. and J. A. Weinberg (1989): "Optimal Contracts with Costly State Falsification," *Journal of Political Economy*, 97, 1345–1363.

Landier, A. and G. Plantin (2016): "Taxing the Rich," *The Review of Economic Studies*, 84, 1186–1209.

Myerson, R. B. (1991): *Game Theory, Analysis of Conflict*, Harvard University Press.

Rodina, D. (2016): "Information Design and Career Concerns," Tech. rep., Working Paper.

Rodina, D. and J. Farragut (2016): "Inducing Effort through Grades," Tech. rep., Working paper.

Roesler, A.-K. and B. Szentes (2017): “Buyer-Optimal Learning and Monopoly Pricing,” American Economic Review, forthcoming.

Rosar, F. (2017): “Test design under voluntary participation,” *Games and Economic Behavior*, 104, 632–655.

C Online Appendix: General Cost Functions

Here, we take back the analysis of optimal design with costly falsification in Section 8 right before introducing the class of linear cost functions. In particular, we consider cost functions $c(p_B)$ defined on I that satisfy Assumption 3.

If (FI) does not hold, the natural intuition is to proceed as in the case without cost. However, the solution of the indifference differential equation with the original cost function may, in general, not be in Δ^B. To circumvent this problem, we work with a modified cost function such that the differential equation always yields a solution in Δ^B, and this solution is optimal for the problem with the original cost function. We obtain this modified cost function recursively. To understand this, it is useful to rewrite the program of the principal as follows. First, note that Lemma 4 holds with costs, so we can focus on tests $\mathcal{H}$ that are linear to the left of $\hat{\mu}$. Such tests can be parameterized by the slope $\kappa \in \left[1-\mu_0/\hat{\mu}, 1-\mu_0\right]$ of the test to the left of $\hat{\mu}$. Then, we have $H(\hat{\mu}) = \mathcal{H}(\hat{\mu})/\hat{\mu} = \kappa$. And, letting Δ^B_κ denote the set of these tests with slope κ to the left of $\hat{\mu}$, we can rewrite the program of the principal as

$$\max_{\kappa\in[\kappa^*,1-\mu_0]} \max_{\mathcal{H}\in\Delta^B_\kappa} \kappa\hat{\mu}$$

$$\text{s.t.} \quad \hat{\mu} c\left(\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}\right) \geq \kappa\hat{\mu} + \frac{\mu-\hat{\mu}}{\mu-\mu_0}\mathcal{H}(\mu) - \mu H(\mu), \qquad \forall \mu \geq \hat{\mu}. \tag{IC$_0'^c$}$$

Note that the optimal no-cost test $\mathcal{H}^*$ satisfies the no-falsification incentive constraint (IC$_0^c$), so the principal can ensure a payoff above $\mathcal{H}^*(\hat{\mu}) = \kappa^*\hat{\mu}$, which is why we limited the range of slopes over which the principal optimizes to $\left[\kappa^*, 1-\mu_0\right]$.

Next, we show that the cost function can be modified in (IC$_0'^c$) without modifying the constraint it puts on all tests in Δ^B_κ. To understand the intuition behind this modification, recall that (IC$_0'^c$) simply expresses that the net profit from falsification should be lower than the cost, that is $\Pi(\mu) - \Pi(\hat{\mu}) \leq c\left(\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}\right)$. Thus, higher cost helps the principal achieve better outcomes as they enlarge the set of tests that satisfy the no falsification incentive constraint. However, excessively high costs are unnecessary. To see that consider two falsification levels $p_B < p_B'$ in I that induce thresholds $\mu < \mu'$. Then, we show that the difference in net profits between these two falsification levels, $\Pi(\mu') - \Pi(\mu)$, can be bounded above by $\kappa(p_B' - p_B)$ for all tests in Δ^B_κ. Therefore any cost in excess of $c(p_B) + \kappa(p_B' - p_B)$ at p_B' is superfluous, and can be eliminated without any harm to the principal.

This intuition leads us to define the modified cost functions on I by

$$\hat{c}_\kappa(x) = \min_{y\in[0,x]} c(y) + \kappa(x-y).$$

As stated in the following lemma, working with these modified cost functions is without loss of generality because, due to the intuition outlined above, it leads to an equivalent set of incentive constraints. The proof of the lemma consists in deriving the upper bound that we used in the intuition.

Lemma 6. *Suppose that $\mathcal{H} \in \Delta^B_\kappa$. Then $\mathcal{H}$ satisfies (IC$_0'^c$) if and only if it satisfies the same incentive constraint with $\hat{c}_\kappa$, that is*

$$\hat{\mu}\hat{c}_\kappa\left(\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}\right) \geq \kappa\hat{\mu} + \frac{\mu-\hat{\mu}}{\mu-\mu_0}\mathcal{H}(\mu) - \mu H(\mu), \qquad \forall \mu \geq \hat{\mu}. \tag{IC$_0'^{\hat{c}_\kappa}$}$$

Proof. Consider two falsification levels $p'_B > p_B$ in I. Let $\mu' > \mu$ be the thresholds they induce in $\left[\hat{\mu}, 1\right]$. The difference in net profits between these two levels of falsifications is given by

$$\Pi(\mu') - \Pi(\mu) = \left\{\frac{\mu' - \hat{\mu}}{\hat{\mu}(\mu' - \mu_0)}\mathcal{H}(\mu') - \frac{\mu'}{\hat{\mu}}H(\mu')\right\} - \left\{\frac{\mu - \hat{\mu}}{\hat{\mu}(\mu - \mu_0)}\mathcal{H}(\mu) - \frac{\mu}{\hat{\mu}}H(\mu)\right\}.$$

By convexity, $\mathcal{H}(\mu)$ is absolutely continuous, and so is the function $\mu \mapsto \frac{\mu - \hat{\mu}}{\mu - \mu_0}$, therefore we can write the difference between the first terms in each bracket as

$$\begin{aligned}\frac{\mu' - \hat{\mu}}{\hat{\mu}(\mu' - \mu_0)}\mathcal{H}(\mu') - \frac{\mu - \hat{\mu}}{\hat{\mu}(\mu - \mu_0)}\mathcal{H}(\mu) &= \int_{\mu}^{\mu'} \left\{\frac{\hat{\mu} - \mu_0}{\hat{\mu}(x - \mu_0)^2}\mathcal{H}(x) + \frac{(x - \hat{\mu})}{\hat{\mu}(x - \mu_0)}H(x)\right\}dx \\ &= \int_{\mu}^{\mu'} \frac{\hat{\mu} - \mu_0}{\hat{\mu}(x - \mu_0)^2}\left\{\mathcal{H}(x) - (x - \mu_0)H(x)\right\}dx,\end{aligned}$$

Then, by convexity, we have

$$\frac{\int_{\mu}^{\mu'} H(x)dx}{\mu' - \mu} = \frac{\mathcal{H}(\mu') - \mathcal{H}(\mu)}{\mu' - \mu} \leq H(\mu')$$

implying

$$-\frac{1}{\hat{\mu}}\int_{\mu}^{\mu'} H(x)dx \geq -\frac{\mu'}{\hat{\mu}}H(\mu') + \frac{\mu}{\hat{\mu}}H(\mu') \geq -\frac{\mu'}{\hat{\mu}}H(\mu') + \frac{\mu}{\hat{\mu}}H(\mu).$$

Reassembling everything, we have

$$\begin{aligned}\Pi(\mu') - \Pi(\mu) &\leq \int_{\mu}^{\mu'} \frac{\hat{\mu} - \mu_0}{\hat{\mu}(x - \mu_0)^2}\left\{\mathcal{H}(x) - (x - \mu_0)H(x)\right\}dx \\ &\leq \left\{\mathcal{H}(\hat{\mu}) - (\hat{\mu} - \mu_0)H(\hat{\mu})\right\}\int_{\mu}^{\mu'} \frac{\hat{\mu} - \mu_0}{\hat{\mu}(x - \mu_0)^2}dx \\ &\leq \mu_0\kappa\left\{\frac{\mu' - \hat{\mu}}{\hat{\mu}(\mu' - \mu_0)} - \frac{\mu - \hat{\mu}}{\hat{\mu}(\mu - \mu_0)}\right\} \\ &\leq \kappa\left\{\frac{\mu_0(\mu' - \hat{\mu})}{\hat{\mu}(\mu' - \mu_0)} - \frac{\mu_0(\mu - \hat{\mu})}{\hat{\mu}(\mu - \mu_0)}\right\} = \kappa\left(p'_B - p_B\right),\end{aligned}$$

where the second inequality is implied by Lemma 5, the third line is due to the linearity of $\mathcal{H}$ to the left of $\hat{\mu}$, which yields $\mathcal{H}(\hat{\mu}) = \hat{\mu}H(\hat{\mu}) = \kappa\hat{\mu}$.

□

The modified cost function satisfies the following technical properties which are crucial in proving that the solution to the differential equation with the modified cost function is in Δ^B.

Lemma 7. *For every $\kappa \in \left[\kappa^*, 1 - \mu_0\right]$, the modified cost function $\hat{c}_\kappa(x)$ is well defined, absolutely continuous, nonnegative and nondecreasing on I. It satisfies $\hat{c}_\kappa(0) = 0$, and $\hat{c}_\kappa(x) \leq \min\{\kappa x, c(x)\}$ for every $x \in I$. Furthermore, $\kappa x - \hat{c}_\kappa(x)$ is nondecreasing, and, for $\kappa' > \kappa$, $\hat{c}_{\kappa'}(x) \geq \hat{c}_\kappa(x)$ for every $x \in I$.*

Proof. $\hat{c}_\kappa(\cdot)$ is well defined since the function $y \mapsto c(y) + \kappa(y - x)$ is continuous and therefore admits a minimum on $[0, x]$. $\hat{c}_\kappa(x)$ is nonnegative as the minimum of a nonnegative function.

By definition, $\hat{c}(x) \leq c(0) + \kappa(x-0) = \kappa x$, and $\hat{c}(x) \leq c(x)$. This implies $\hat{c}(0) = 0$. Let $\hat{y}_\kappa(x) = \arg\min_{y\in[0,x]} c(y) + \kappa(x-y)$. By the maximum theorem, $\hat{y}(\cdot)$ is a nonempty valued correspondence. Consider $x' > x$, and $y' \in \hat{y}_\kappa(x')$. Suppose first that $y' > x$. Then

$$\hat{c}_\kappa(x') = c(y') + \kappa(x'-y') \geq c(y') \geq c(x) \geq \hat{c}_\kappa(x).$$

Suppose, otherwise, that $y' \leq x$. Then

$$\hat{c}_\kappa(x') = c(y') + \kappa(x-y') + \kappa(x'-x) \geq \hat{c}_\kappa(x) + \kappa(x'-x) \geq \hat{c}_\kappa(x).$$

Hence $\hat{c}_\kappa(\cdot)$ is nondecreasing. Next, let $y \in \hat{y}_\kappa(x)$, and note that

$$\hat{c}_\kappa(x') - \hat{c}_\kappa(x) \leq \big[c(y) + \kappa(x'-y)\big] - \big[c(y) - \kappa(x-y)\big] \leq \kappa(x'-x).$$

Therefore, $\hat{c}_\kappa(\cdot)$ is κ-Lipschitz continuous, and in particular absolutely continuous. Furthermore, this implies that $\kappa x - \hat{c}_\kappa(x)$ is nondecreasing.

Next, for $\kappa' > \kappa$, and $y' \in \hat{y}_{\kappa'}(x)$, we have

$$\hat{c}_{\kappa'}(x) = c(y') + \kappa'(x-y') \geq c(y') + \kappa(x-y') \geq \hat{c}_\kappa(x).$$

□

In what follows, to simplify notations, we also write the modified cost functions as a function of the induced threshold

$$\gamma_\kappa(\mu) = \hat{c}_\kappa\left(\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}\right).$$

Then, Lemma 6 implies that we can reformulate the program of the principal as

$$\max_{\kappa\in[\kappa^*,1-\mu_0]} \max_{\mathcal{H}\in\Delta_\kappa^B} \kappa\hat{\mu}$$
$$\text{s.t.} \quad \hat{\mu}\gamma_\kappa(\mu) \geq \kappa\hat{\mu} + \frac{\mu-\hat{\mu}}{\mu-\mu_0}\mathcal{H}(\mu) - \mu H(\mu), \qquad \forall\mu \geq \hat{\mu}.$$

To apply the same idea as in the no-cost case, we would solve the differential equation

$$\hat{\mu}\gamma_\kappa(\mu) = \kappa\hat{\mu} + \frac{\mu-\hat{\mu}}{\mu-\mu_0}\mathcal{H}(\mu) - \mu H(\mu)$$

with initial conditions $H(\hat{\mu}) = \mathcal{H}(\hat{\mu})/\hat{\mu} = \kappa$, and then set κ so that $\mathcal{H}(1) = 1-\mu_0$. The problem with directly applying this idea is that it leads to a very intractable equation in κ making it difficult to characterize the solution. Furthermore, it is difficult to assess existence or uniqueness of a solution, and even more so, to show that a solution is indeed a test. Therefore, we adopt a different method that characterizes the solution of the principal's problem recursively as follows.

- $\kappa_0 = 1 - \mu_0$.
- To get κ_{n+1}, we write the following linear differential equation on $\big[\hat{\mu}, 1\big]$

$$H(\mu) - \frac{\mu-\hat{\mu}}{\mu(\mu-\mu_0)}\mathcal{H}(\mu) = \frac{\hat{\mu}}{\mu}\big(\kappa - \gamma_{\kappa_n}(\mu)\big),$$

with initial conditions $H(\hat{\mu}) = \mathcal{H}(\hat{\mu})/\hat{\mu} = \kappa$. The solution is then given by

$$\mathcal{H}(\mu) = \hat{\mu}\psi(\mu)\left[\kappa\left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx\right) - \int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_n}(x)}{x\psi(x)}dx\right],$$

and we set κ_{n+1} to be the unique value of κ such that $\mathcal{H}(1) = 1 - \mu_0$. That is, we have the following recurrence equation

$$\kappa_{n+1} = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \int_{\hat{\mu}}^{1} \frac{\gamma_{\kappa_n}(x)}{x\psi(x)}dx\right)\left(1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx\right)^{-1}. \tag{REC}$$

Finally, we let $\mathcal{H}_n(\mu)$ be the solution to the differential equation with $\kappa = \kappa_{n+1}$.

We show in the next theorem that this sequence always converges, and we can therefore define a limit to the sequence of functions $\mathcal{H}_n$. If the limit of this sequence is a test, that is, if it lies in Δ^B, then it is optimal. However, we need to make another assumption on the cost function to ensure that it is the case.[19]

Assumption 4. *The function $\frac{c(p_B)}{p_B}$ is nonincreasing on I.*

Then, we have the following theorem.

Theorem 4. *If the cost function satisfies (FI), then the optimal test is the fully informative one. Otherwise, the sequence $\{\kappa_n\}$ is decreasing and admits a limit $\kappa_c^* \in (\kappa^*, 1-\mu_0)$. Then, the function given by*

$$\mathcal{H}_c^*(\mu) = \begin{cases} \kappa_c^*\mu & \text{if } \mu \le \hat{\mu} \\ \hat{\mu}\psi(\mu)\left[\kappa_c^*\left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx\right) - \int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx\right] & \text{if } \mu \ge \hat{\mu} \end{cases}$$

is an optimal test whenever the cost function satisfies Assumption 4. Furthermore, any other optimal experiment must be linear to the left of $\hat{\mu}$ and less informative than $\mathcal{H}_c^*$. Finally, for all $\mu \in (0,1)$, $\mathcal{H}_{FI}(\mu) > \mathcal{H}_c^*(\mu) > \mathcal{H}^*(\mu)$. If Assumption 4 is not satisfied, then κ_c^* is an upper bound on the modified payoff of the principal.

Proof. We have already proved the first point. Suppose, therefore that the cost function does not satisfy (FI). We prove the results in the theorem in several steps.

Step 1: convergence of the sequence $\{\kappa_n\}$. To show that the sequence $\{\kappa_n\}$ is decreasing, we proceed by induction. First, note that when the cost function is given by $(1-\mu_0)p_B$, the fully informative test makes the incentive constraint of the agent hold with equality at ever $\mu \ge \hat{\mu}$. Therefore, the fully informative test solves the linear differential equation

$$H(\mu) - \frac{\mu - \hat{\mu}}{\mu(\mu - \mu_0)}\mathcal{H}(\mu) = \frac{\hat{\mu}}{\mu}\left(1 - \mu_0 - (1-\mu_0)\frac{\mu_0(\mu - \hat{\mu})}{\hat{\mu}(\mu - \mu_0)})\right),$$

implying that we have, for all $\mu \ge \hat{\mu}$,

$$(1-\mu_0)\mu = \mathcal{H}_{FI}(\mu) = \hat{\mu}\psi(\mu)\left[\kappa_0\left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx\right) - \int_{\hat{\mu}}^{\mu} \frac{\kappa_0\frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)})}{x\psi(x)}dx\right],$$

[19] Note that Assumption 4 implies Assumption 3.

and, in particular, at $\mu = 1$

$$\kappa_0 = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \int_{\hat{\mu}}^{1} \frac{\kappa_0 \frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)}}{x\psi(x)} dx \right) \left(1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)} dx \right)^{-1}.$$

By construction, κ_1 is given by

$$\kappa_1 = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \int_{\hat{\mu}}^{1} \frac{\gamma_{\kappa_0}(x)}{x\psi(x)} dx \right) \left(1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)} dx \right)^{-1}.$$

By Lemma 7, we have

$$\gamma_{\kappa_0}(x) = \hat{c}_{\kappa_0}\left(\frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)} \right) \leq \min \left\{ \kappa_0 \frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)}, c\left(\frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)} \right) \right\}.$$

Then, $\gamma_{\kappa_0}(x) \leq \kappa_0 \frac{\mu_0(x-\hat{\mu})}{\hat{\mu}(x-\mu_0)}$ for all $x \in \left[\hat{\mu}, 1\right]$, and because $c(\cdot)$ does not satisfy (FI), and is continuous, there exists an open interval over which the inequality is strict. Therefore, we must have $\kappa_1 < \kappa_0 = 1 - \mu_0$.

Next, suppose that for $n \geq 1$, we have $\kappa_n \leq \kappa_{n-1}$. Then, Lemma 7 implies that we have $\gamma_{\kappa_n}(x) \leq \gamma_{\kappa_{n-1}}(x)$, for all $x \in \left[\hat{\mu}, 1\right]$, and therefore, by (REC), $\kappa_{n+1} \leq \kappa_n$.

Next, note the definition of κ^* implies that, for all $n \geq 0$, $\kappa_n > \kappa^*$. $\{\kappa_n\}$ is therefore a decreasing sequence bounded from below, hence it must converge to a limit $\kappa_c^* \in \left[\kappa^*, 1-\mu_0\right)$. Furthermore, κ_c^* must be a fixed point of the recurrence equation (REC). Therefore

$$\kappa_c^* = \left(\frac{1-\mu_0}{\hat{\mu}\psi(1)} + \int_{\hat{\mu}}^{1} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)} dx \right) \left(1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)} dx \right)^{-1},$$

and, since $\kappa_c^* > 0$, $\gamma_{\kappa_c^*}(x) > 0$, for all $x > \hat{\mu}$, implying that $\kappa_c^* > \kappa^*$.

Step 2: $\mathcal{H}_{FI}(\mu) > \mathcal{H}_c^*(\mu) > \mathcal{H}^*(\mu)$. Using the expressions of $\mathcal{H}^*$ and $\mathcal{H}_c^*$, we can write the difference of the two functions for each $\mu \geq \hat{\mu}$ as

$$\begin{aligned}\mathcal{H}_c^*(\mu) - \mathcal{H}^*(\mu) = & \frac{\hat{\mu}\int_{\hat{\mu}}^{1} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx}{1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx} \psi(\mu) \int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)} dx \\ & \times \underbrace{\left\{ \frac{1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx} - \frac{1 + \int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{1} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx} \right\}}_{\equiv \Delta(\mu)},\end{aligned} \tag{11}$$

where the second equality is from the proof of Theorem 1.

Note that we have $\Delta(1) = 0$. To assess the sign of this term, we compute its derivative

$$\Delta'(\mu) = \left(\frac{1}{\int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx} \right)^2 \frac{1}{\mu\psi(\mu)} \left\{ \int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)} dx - \gamma_{\kappa_c^*}(\mu) \left(1 + \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)} \right) \right\}.$$

Since $\gamma_{\kappa_c^*}(\cdot)$ is nondecreasing, we have $\int_{\hat{\mu}}^{\mu} \frac{\gamma_{\kappa_c^*}(x)}{x\psi(x)}dx \leq \gamma_{\kappa_c^*}(\mu) \int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx$, and therefore, $\Delta'(\mu) < 0$ on $\big(\hat{\mu}, 1\big]$, implying that $\mathcal{H}_c^*(\mu) > \mathcal{H}^*(\mu)$ on $\big[\hat{\mu}, 1\big)$, which easily extends to $\big(0, \hat{\mu}\big]$ by linearity of both functions on this interval and continuity at $\hat{\mu}$.

Next, note that the fully informative test $\mathcal{H}_{FI}$ is the solution of the differential equation with cost when the cost function is given by $\gamma_{FI}(\mu) = (1-\mu_0)\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)}$. Hence, we can write the following version of (11),

$$\begin{aligned}\mathcal{H}_{FI}(\mu) - \mathcal{H}^*(\mu) = &\frac{\hat{\mu}\int_{\hat{\mu}}^{1} \frac{\gamma_{FI}(x)}{x\psi(x)}dx}{1+\int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx}\psi(\mu)\int_{\hat{\mu}}^{\mu} \frac{\gamma_{FI}(x)}{x\psi(x)}dx \\ &\times \left\{ \frac{1+\int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{\mu} \frac{\gamma_{FI}(x)}{x\psi(x)}dx} - \frac{1+\int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{1} \frac{\gamma_{FI}(x)}{x\psi(x)}dx} \right\}. \qquad (12)\end{aligned}$$

Subtracting (11) from (12)

$$\begin{aligned}\mathcal{H}_{FI}(\mu) - \mathcal{H}_c^*(\mu) = &\frac{\hat{\mu}\int_{\hat{\mu}}^{1} \frac{\delta(x)}{x\psi(x)}dx}{1+\int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx}\psi(\mu)\int_{\hat{\mu}}^{\mu} \frac{\delta(x)}{x\psi(x)}dx \\ &\times \underbrace{\left\{ \frac{1+\int_{\hat{\mu}}^{\mu} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{\mu} \frac{\delta(x)}{x\psi(x)}dx} - \frac{1+\int_{\hat{\mu}}^{1} \frac{1}{x\psi(x)}dx}{\int_{\hat{\mu}}^{1} \frac{\delta(x)}{x\psi(x)}dx} \right\}}_{\equiv\tilde{\Delta}(\mu)}, \qquad (13)\end{aligned}$$

where $\delta(x) = \gamma_{FI}(x) - \gamma_{\kappa_c^*}(x)$ is bounded below by 0, above by $\gamma_{FI}(x)$. Lemma 7 implies that $\delta(x)$ is non decreasing in x. Therefore, applying the same argument as for Δ, we can show that $\mathcal{H}_{FI}(\mu) > \mathcal{H}_c^*(\mu)$ on $\big(0, 1\big)$.

Step 3: $\mathcal{H}_c^* \in \Delta^B$: Next, we show that $\mathcal{H}_c^* \in \Delta^B$. Given that we already have $\mathcal{H}_{FI}(\mu) > \mathcal{H}_c^*(\mu) > \mathcal{H}^*(\mu)$, it is sufficient to show that $\mathcal{H}_c^*$ is convex to ensure that it is in Δ^B. Using the same computations as in the case without cost, we can write

$$\begin{aligned}\mathcal{H}_c^*(\mu) = \kappa_c^*(\mu-\mu_0)&\left\{1+\mu_0(\hat{\mu}-\mu_0)^{\frac{\hat{\mu}}{\mu_0}-1}\hat{\mu}^{-\frac{\hat{\mu}}{\mu_0}}\left(\frac{\mu}{\mu-\mu_0}\right)^{\frac{\hat{\mu}}{\mu_0}}\right\} \\ &- (\mu-\mu_0)^{1-\frac{\hat{\mu}}{\mu_0}}\mu^{\frac{\hat{\mu}}{\mu_0}}\hat{\mu}\int_{\hat{\mu}}^{\mu} \gamma_{\kappa_c^*}(x)(x-\mu_0)^{\frac{\hat{\mu}}{\mu_0}-1}x^{-\frac{\hat{\mu}}{\mu_0}-1}dx.\end{aligned}$$

We introduce the function

$$\varphi_\kappa(\mu) = \kappa p_B - \hat{c}_\kappa(p_B) = \kappa\frac{\mu_0(\mu-\hat{\mu})}{\hat{\mu}(\mu-\mu_0)} - \gamma_\kappa(\mu).$$

By Lemma 7, this function is nonnegative and nondecreasing in p_B, and hence in μ. Then, we can rewrite $\mathcal{H}_c^*$ as follows

$$\begin{aligned}\mathcal{H}_c^*(\mu) = \kappa_c^*\mu + (\mu-\mu_0)^{1-\frac{\hat\mu}{\mu_0}}\mu^{\frac{\hat\mu}{\mu_0}}\hat\mu\Bigg\{ &\frac{\kappa\mu_0}{\hat\mu}\underbrace{\left(\hat\mu^{-\frac{\hat\mu}{\mu_0}}(\hat\mu-\mu_0)^{\frac{\hat\mu}{\mu_0}-1} - \mu^{-\frac{\hat\mu}{\mu_0}}(\mu-\mu_0)^{\frac{\hat\mu}{\mu_0}-1}\right)}_{=\int_{\hat\mu}^{\mu}(x-\hat\mu)(x-\mu_0)^{\frac{\hat\mu}{\mu_0}-2}x^{-\frac{\hat\mu}{\mu_0}-1}dx} \\ &- \int_{\hat\mu}^{\mu}\gamma_{\kappa_c^*}(x)(x-\mu_0)^{\frac{\hat\mu}{\mu_0}-1}x^{-\frac{\hat\mu}{\mu_0}-1}dx.\Bigg\}\end{aligned}$$

Therefore

$$\mathcal{H}_c^*(\mu) = \kappa_c^*\mu + (\mu-\mu_0)^{1-\frac{\hat\mu}{\mu_0}}\mu^{\frac{\hat\mu}{\mu_0}}\hat\mu\int_{\hat\mu}^{\mu}\varphi_{\kappa_c^*}(x)(x-\mu_0)^{\frac{\hat\mu}{\mu_0}-1}x^{-\frac{\hat\mu}{\mu_0}-1}dx. \tag{14}$$

Differentiating, we get

$$H_c^*(\mu) = \kappa_c^* + \hat\mu\left\{\frac{\varphi_{\kappa_c^*}(\mu)}{\mu} + (\mu-\hat\mu)(\mu-\mu_0)^{-\frac{\hat\mu}{\mu_0}}\mu^{\frac{\hat\mu}{\mu_0}-1}\int_{\hat\mu}^{\mu}\varphi_{\kappa_c^*}(x)(x-\mu_0)^{\frac{\hat\mu}{\mu_0}-1}x^{-\frac{\hat\mu}{\mu_0}-1}dx\right\}. \tag{15}$$

Note that this implies that $H_c^*(\mu) \geq \kappa_c^*$ for all $\mu \geq \hat\mu$. Next, note that, by definition, the function $\mathcal{H}_c^*$ solves the differential equation

$$\frac{\mu-\hat\mu}{\mu-\mu_0}\mathcal{H}_c^*(\mu) - \mu H_c^*(\mu) + \kappa_c^*\hat\mu = \hat\mu\gamma_{\kappa_c^*}(\mu),$$

which we can also write

$$\frac{\mu-\hat\mu}{\mu-\mu_0}\left(\mathcal{H}_c^*(\mu) - (\mu-\mu_0)H_c^*(\mu)\right) - \hat\mu\left(H_c^*(\mu) - \kappa_c^*\right) = \hat\mu\gamma_{\kappa_c^*}(\mu).$$

Differentiating this equation, we obtain

$$\begin{aligned}\mu h_c^*(\mu) &= \frac{\hat\mu-\mu_0}{(\mu-\mu_0)^2}\left(\mathcal{H}_c^*(\mu) - (\mu-\mu_0)H_c^*(\mu)\right) - \hat\mu\gamma'_{\kappa_c^*}(\mu) \\ &= \frac{\hat\mu-\mu_0}{(\mu-\mu_0)(\mu-\hat\mu)}\left\{H_c^*(\mu) - \kappa + \gamma_{\kappa_c^*}(\mu) - \frac{(\mu-\mu_0)(\mu-\hat\mu)}{\hat\mu-\mu_0}\gamma'_{\kappa_c^*}(\mu)\right\} \\ &= \frac{\hat\mu-\mu_0}{(\mu-\mu_0)(\mu-\hat\mu)}\left\{H_c^*(\mu) - \kappa_c^* + \hat c_{\kappa_c^*}(p_B) - p_B\hat c'_{\kappa_c^*}(p_B)\right\}\end{aligned}$$

We have already proved that $H_c^*(\mu) - \kappa_c^* \geq 0$, and it is easy to see that Assumption 4 implies that $\hat c_{\kappa_c^*}(p_B)/p_B$ is nonincreasing, and therefore $\hat c_{\kappa_c^*}(p_B) - p_B\hat c'_{\kappa_c^*}(p_B) \geq 0$.

Step 4: Optimality of $\mathcal{H}_c^*$: Let $\mathcal{H} \in \Delta_B$ be a test with $\mathcal{H}(\hat\mu) = \hat\mu\kappa'$, and $\kappa' > \kappa_c^*$ that satisfies the no-falsification incentive constraint. By Lemma 4, we can take this test to be linear to the left of $\hat\mu$, that is $\mathcal{H} \in \Delta_\kappa^B$. Then $\mathcal{H}$ satisfies, for every $\mu \geq \hat\mu$,

$$\hat\mu\gamma_{\kappa'}(\mu) \geq \kappa'\hat\mu + \frac{\mu-\hat\mu}{\mu-\mu_0}\mathcal{H}(\mu) - \mu H(\mu).$$

Since $\kappa' > \kappa_c^*$, there must exist some $n \geq 0$ such that $\kappa_n \geq \kappa' > \kappa_{n+1}$. Then, by Lemma 7, $\gamma_{\kappa_n}(\mu) \geq \gamma_{\kappa'}(\mu)$, implying that the no-falsification incentive constraint must hold with γ_{κ_n} as well, that is, for every $\mu \geq \hat{\mu}$,

$$\hat{\mu}\gamma_{\kappa_n}(\mu) \geq \kappa'\hat{\mu} + \frac{\mu - \hat{\mu}}{\mu - \mu_0}\mathcal{H}(\mu) - \mu H(\mu).$$

Next consider the function $\mathcal{H}_n(\mu)$, which, by definition, satisfies $\mathcal{H}_n(\hat{\mu}) = \hat{\mu}\kappa_{n+1}$, and $\mathcal{H}_n(1) = 1 - \mu_0$, and, for every $\mu \geq \hat{\mu}$,

$$\hat{\mu}\gamma_{\kappa_n}(\mu) = \kappa_{n+1}\hat{\mu} + \frac{\mu - \hat{\mu}}{\mu - \mu_0}\mathcal{H}_n(\mu) - \mu H_n(\mu).$$

Since $\mathcal{H}(\hat{\mu}) > \mathcal{H}_n(\hat{\mu})$, and $\mathcal{H}(1) = \mathcal{H}_n(1) = 1 - \mu_0$, there exists some $\tilde{\mu} \in \big(\hat{\mu}, 1\big]$, such that $\mathcal{H}(\tilde{\mu}) = \mathcal{H}_n(\tilde{\mu})$, and $\mathcal{H}(\mu) > \mathcal{H}_n(\mu)$ for $\mu \in \big[\hat{\mu}, \tilde{\mu}\big)$. But then, we must have $H(\tilde{\mu}) \leq H_n(\tilde{\mu})$. Therefore

$$\begin{aligned}\hat{\mu}\gamma_{\kappa_n}(\tilde{\mu}) &\geq \kappa'\hat{\mu} + \frac{\tilde{\mu} - \hat{\mu}}{\tilde{\mu} - \mu_0}\mathcal{H}(\tilde{\mu}) - \tilde{\mu}H(\tilde{\mu}) \\ &> \kappa_{n+1}\hat{\mu} + \frac{\tilde{\mu} - \hat{\mu}}{\tilde{\mu} - \mu_0}\mathcal{H}_n(\tilde{\mu}) - \tilde{\mu}H_n(\tilde{\mu}) = \hat{\mu}\gamma_{\kappa_n}(\tilde{\mu}),\end{aligned}$$

a contradiction. □

Thus, our recursive approach delivers the optimal test whenever Assumption 4 is satisfied. When Assumption 4 is not satisfied, the recursive approach still delivers a limit function $\mathcal{H}_c^*$. However, we cannot ensure that this function is convex, and therefore corresponds to a test. But it is still true that any optimal test $\mathcal{H}(\mu)$ must lie below $\mathcal{H}_c^*(\mu)$, and therefore the modified payoff of the principal is bounded above by $\mathcal{H}_c^*(\hat{\mu}) = \kappa_c^*\hat{\mu}$. Furthermore, for any cost function, if $\mathcal{H}_c^*$ happens to be convex so that it is a test, then it is an optimal test.