Altruism in Networks
Supplementary Appendix

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This supplementary material contains proofs and additional results that complement the paper "Altruism in Networks". The first section studies the transfer cost minimization problem that underlies the potential maximization problem of the main paper, and uses it to derive additional properties of equilibrium transfers, and provide different proofs of some results. The generic uniqueness of equilibrium transfers is a consequence of this analysis. We also use this section to explain the connections between our altruistic transfer game and two classical transportation problems: the minimum cost flow problem, and the Monge-Kantorovich optimal transportation problem. In the second section, we show convergence of best response dynamics in the transfer game. In the third section, we look at conditions for the presence or absence of transfer intermediaries, and provides a proof of Theorem 2 of the paper. Finally, in the fourth section, we consider comparative statics with respect to initial income profiles and altruism. We prove the genericity result used in the proofs of the paper, and provide some additional comparative statics results.

APPENDIX A : The Cost Minimization Approach

In this section, we analyze in detail the cost minimization problem and use it to draw connection with classical linear programming problems, and to prove generic uniqueness of equilibrium transfers. We also exhibit some additional properties of optimal transfer networks such as cyclical monotonicity. The presentation of the first results borrows from Galichon (2011). For an overview of the use of optimal transport methods in economics see Galichon (2016).

Recall that the maximization of the potential is related to the cost minimization problem

$$c(\mathbf{y}^0, \mathbf{y}) = \min_{\mathbf{T} \in S(\mathbf{y})} \sum_{(i,j) \in A} c_{ij} t_{ij}, \tag{MCF}$$

where $A = \{(i,j) : \alpha_{ij} > 0\}$ is the set of arcs of the altruistic network $\boldsymbol{\alpha}$, and $S(\mathbf{y}) = \{\mathbf{T} \in S : \forall i, y_i = y_i^0 - \sum_j t_{ij} + \sum_j t_{ji}\}$ is a closed convex polytope since it is defined by a finite number of weak inequalities. Note that $S(\mathbf{y})$ is unbounded if the altruistic network $\boldsymbol{\alpha}$ admits a directed cycle, since one can then indefinitely increase the transfers of any $\mathbf{T} \in S(\mathbf{y})$ along the cycle while still reaching $\mathbf{y}$ from $\mathbf{y}^0$. This problem is a classical linear programming problem known as the Minimum Cost Flow problem. Indeed, if each c_{ij} is interpreted as the marginal transportation cost between i and j, this problem consists of minimizing transportation cost over the network of agents with the constraint of reaching distribution $\mathbf{y}$ from distribution $\mathbf{y}^0$.

In network flow problems, a transfer profile $\mathbf{T}$ is called a *flow*, and we will sometimes use this terminology. A first useful result from the network flow literature (see Galichon, 2011) says that any flow can be decomposed into paths and cycles. Before we do that, we partition the set of agents into three sets: the sets of net givers $I_G = \{i : y_i < y_i^0\}$ (or *sources*), the set of net receivers $I_R = \{i : y_i > y_i^0\}$ (or *sinks*), and the remaining agents. We let $\mathcal{P}_{ij}$ be the set of paths between i and j in the altruistic network, $\mathcal{P} = \bigcup_{(i,j)\in I_G \times I_R} \mathcal{P}_{ij}$, be the set of paths from net givers to net receivers, and $\mathcal{C}$ be the set of cycles in the network. For any $\rho \in \mathcal{P}$, let h_ρ be the intensity of the flow along ρ, and for each $\gamma \in \mathcal{C}$, let g_γ be the intensity of the flow along γ. A flow on a path is called a path flow, and a flow on a cycle is called a cycle flow. Together, the vectors $\mathbf{h}$ and $\mathbf{g}$ define a feasible transfer profile $\mathbf{T}$ through the equation

$$t_{ij} = \sum_{\rho \in \mathcal{P}} h_\rho \mathbf{1}_{(i,j)\in\rho} + \sum_{\gamma\in\mathcal{C}} g_\gamma \mathbf{1}_{(i,j)\in\gamma}. \tag{A.1}$$

If in addition,

$$\forall i \in I_G, \ \sum_{j\in I_R} \sum_{\rho\in\mathcal{P}_{ij}} h_\rho = y_i^0 - y_i, \tag{A.2}$$

and

$$\forall j \in I_R, \ \sum_{i\in I_G} \sum_{\rho\in\mathcal{P}_{ij}} h_\rho = y_j - y_j^0, \tag{A.3}$$

then $\mathbf{T} \in S(\mathbf{y})$. The following proposition shows that every transfer profile can be decomposed in such a way. However, it is easy to see that this decomposition is not necessarily unique. The proof we provide here is adapted from Galichon (2011).

Proposition A.1 (Flow Decomposition) *Any transfer plan* $\mathbf{T} \in S(\mathbf{y})$ *can be decomposed into path flows and cycle flows of intensities* $\mathbf{h}$ *and* $\mathbf{g}$ *according to* (A.1)*, and such that* $\mathbf{h}$ *satisfies* (A.2) *and* (A.3)*. Conversely any distribution of path flows and cycle flows of intensities* $\mathbf{h}$ *and* $\mathbf{g}$ *such that* $\mathbf{h}$ *satisfies* (A.2) *and* (A.3) *defines a transfer plan* $\mathbf{T} \in S(\mathbf{y})$ *through equation* (A.1)*.*

Proof. The second part of the proposition is immediate. For the first part, let $\mathbf{T} \in S(\mathbf{y})$, and consider the following maximization problem

$$\begin{aligned} \max_{\mathbf{h},\mathbf{g}} \; & \sum_{\rho \in \mathcal{P}} h_\rho + \sum_{\gamma \in \mathcal{C}} g_\gamma \\ \text{s.t.} & \sum_{\rho \in \mathcal{P}} h_\rho \mathbf{1}_{(i,j)\in\rho} + \sum_{\gamma \in \mathcal{C}} g_\gamma \mathbf{1}_{(i,j)\in\gamma} \leq t_{i,j}, \quad \forall (i,j) \in A \end{aligned} \tag{P}$$

Because this is a linear program over a bounded set, it has a solution $(\mathbf{h}, \mathbf{g})$. Consider the flow $\mathbf{T}'$, defined by

$$t'_{ij} = \sum_{\rho \in \mathcal{P}} h_\rho \mathbf{1}_{(i,j)\in\rho} + \sum_{\gamma \in \mathcal{C}} g_\gamma \mathbf{1}_{(i,j)\in\gamma} \leq t_{ij}$$

Suppose that this inequality holds strictly. If $(i,j) \in I_G \times I_R$, then one can increase the flow going through the path $\rho = (i,j) \in \mathcal{P}$ by $t_{ij} - t'_{ij}$ while still satisfying the constraint in (P). Since that would strictly improve the objective function of the program (P), that would lead to a contradiction. Suppose for example that $j \notin I_R$. Then there must exist an agent j' such that $t'_{jj'} < t_{jj'}$, for otherwise the conservation equation at j would be violated by $\mathbf{T}'$. Similarly, if $i \notin I_G$, then there exists an agent i' such that $t'_{i'i} < t_{i'i}$. Extending t'_{ij} to the left and the right in this way, it must be the case that we end up with a path $\rho \in \mathcal{P}$ that goes through (i,j), or a cycle $\mu \in \mathcal{C}$ that does not necessarily go through (i,j), and such that each for each (ℓ, k) that belongs to ρ or μ, $t'_{\ell k} < t_{\ell k}$. Then there is some leeway to increase the intensity h_ρ or g_μ, and thus strictly improve the objective of the maximization problem (P), while still satisfying its constraint: a contradiction. ■

Note that if $\mathbf{T}$ has a cycle, it may admit a flow decomposition that puts 0 intensity on all cycles. However, there must be a decomposition with a cycle, as the following result shows.

Lemma A.1 $\mathbf{T}$ *has a cycle if and only if it admits a flow decomposition that puts positive intensity on a cycle.*

Proof. Let $\gamma = (i_0, \cdots, i_\ell)$ be a cycle of $\mathbf{T}$, and let $\tau = \min_{(i,j)\in\gamma} t_{ij} > 0$. Then let $\mathbf{g}$be the cyclic flow that puts intensity $g_\gamma = \tau$ on the cycle γ, and let $\mathbf{T}' = \mathbf{T} - \mathbf{g}$. $\mathbf{T}'$ is a feasible transfer plan that achieves the same distribution as $\mathbf{T}$, and therefore it has a flow decomposition $(\mathbf{h}', \mathbf{g}')$. Then $(\mathbf{h}', \mathbf{g}' + +\mathbf{g})$ is a flow decomposition of $\mathbf{T}$ that puts positive weight on a cycle. The other implication is trivial. ■

We now introduce a different cost minimization problem, known as the Monge-Kantorovich optimal transportation problem, and show how it is related to the initial problem. In the process, we also prove several important properties of the solutions to (MCF). We start by defining the reduced cost vector $\hat{c}$ with elements

$$\hat{c}_{ij} = \min_{\rho \in \mathcal{P}_{ij}} \sum_{(\ell,k)\in\rho} c_{\ell k},$$

for every (i, j). This reduced cost vector is the cost vector associated with the transitive closure $\hat{\boldsymbol{\alpha}}$ of the altruism network. The cost $\hat{c}_{ij}$ is the lowest cost path between i and j. We call such paths *shortest paths.* They correspond to highest altruism paths. Let $\hat{P}_{ij}$ be the set of shortest paths between i and j.

For every $i \in I_G$ let the $G_i = y_i^0 - y_i$ be the amount of money that needs to be transferred away from i, and for every $j \in I_R$, let $R_j = y_j - y_j^0$ be the amount of money that needs to be transferred to j. By construction, $\sum_{i\in I_G} G_i = \sum_{j\in I_R} R_i$. We can view our problem as that of transferring the amount $\sum_{i\in I_G} G_i$ from I_G to I_R in the least costly way. It is natural to express the cost of transportation between $i \in I_G$ and $j \in I_R$ as $\hat{c}_{ij}$. Formally,

$$\begin{aligned} \min_{\tau \in \mathbb{R}_+^{I_G \times I_R}} \quad & \sum_{(i,j) \in I_G \times I_R} \hat{c}_{ij} \tau_{ij} \\ \text{s.t.} \quad & \sum_{j \in I_R} \tau_{ij} = G_i, \quad \forall i \in I_G \\ & \sum_{i \in I_G} \tau_{ij} = R_j, \quad \forall j \in I_R. \end{aligned} \tag{MK}$$

This program is a Monge-Kantorovich optimal transportation problem with discrete source and target distributions. The two problems are related in the following way. Here again, our presentation borrows from Galichon (2011).

Theorem A.1 *If* $\mathbf{T}$ *solves* (MCF)*, then it has no cycles, and all the paths with positive intensity in its flow decomposition are shortest paths. Furthermore, the set of solutions to* (MCF) *is a nonempty, compact and convex polytope. The value* $c(\boldsymbol{y}, \boldsymbol{y}^0)$ *of the cost minimization problem is equal to the value function of* (MK)*. The solutions of* (MCF) *can be obtained from the solutions of* (MK) *by distributing each* τ_{ij} *across the paths in* $\hat{P}_{ij}$*, the shortest paths from* i *to* j*. The solutions of* (MK) *can be obtained from the solutions of* (MCF) *by setting* τ_{ij} *equal to the sum of the intensities over paths in* $\mathcal{P}_{ij}$ *in the flow decomposition of a solution* $\mathbf{T}$.

Proof. Using the flow decomposition theorem, we can rewrite (MCF) as

$$\begin{aligned} \min_{\mathbf{h}, \mathbf{g}} \quad & \sum_{\rho \in \mathcal{P}} h_\rho c_\rho + \sum_{\gamma \in \mathcal{C}} g_\gamma c_\gamma \\ \text{s.t.} \quad & \sum_{j \in I_R} \sum_{\rho \in \mathcal{P}_{ij}} h_\rho = G_i, \quad \forall i \in I_G \\ & \sum_{i \in I_G} \sum_{\rho \in \mathcal{P}_{ij}} h_\rho = R_j, \quad \forall j \in I_R, \end{aligned}$$

where $c_\rho = \sum_{(i,j) \in \rho} c_{ij}$ and $c_\gamma = \sum_{(i,j) \in \gamma} c_{ij}$.

Since cycles cannot help satisfying the constraints, it is optimal to set $g_\gamma = 0$ for every $\gamma \in \mathcal{C}$. Hence optimal transfer networks have no cycle in their flow decomposition, and are therefore acyclic by Lemma A.1. It is also clear that only shortest paths can have strictly positive intensity. Indeed, if there exists a path $\rho \in \mathcal{P}_{ij}$ such that $h_\rho > 0$, and ρ is not a shortest path, then reassigning intensity h_ρ to another path $\rho' \in \hat{\mathcal{P}}_{ij}$ would lead to a cost reduction of $\big(c_\rho - \hat{c}_{ij}\big)h_\rho > 0$.

Having proved that the set of solutions to (MCF) is acyclic, we can solve the minimization problem over the set of acyclic transfer plans. Contrary to the set of transfer plans, it is bounded as no transfer t_{ij} can exceed the total amount of money available $\sum_i y_i^0$. It is easy to show that it is also a closed set, therefore the minimization problem minimizes a continuous function over a compact set, implying the existence of a solution. Since (MCF) is a linear problem, we know that the solution set is a closed convex polytope, and since all solutions are acyclic, it is also bounded and hence compact.

Since optimal transfers only use shortest paths, we can rewrite the objective function of the transformed program as

$$\sum_{(i,j)\in I_G\times I_R}\sum_{\rho\in\hat{\mathcal{P}}_{ij}} h_\rho c_\rho = \sum_{(i,j)\in I_G\times I_R} \hat{c}_{ij} \sum_{\rho\in\hat{\mathcal{P}}_{ij}} h_\rho .$$

Letting $\tau_{ij} = \sum_{\rho\in\hat{\mathcal{P}}_{ij}} h_\rho$, the transformed program becomes (MCF). This shows that the two programs have the same value function, and how to obtain the solutions of (MCF) and (MK) from one another. ■

To get a better understanding of the structure of the set of solutions to (MCF), we start by describing the structure of the set $S(\mathbf{y})$. We know that it is a possibly unbounded convex polytope. Therefore it can be expressed as the convex hull of a finite set of points and directions. We will now characterize its set of extreme points and directions. The directions will be given by the cycles of A. For every cycle $\gamma \in \mathcal{C}$, let $\mathbf{T}^\gamma$ be the flow defined by $t_{ij}^\gamma = \mathbf{1}_{(i,j)\in\gamma}$. To describe the set of extreme points, we need some additional notations. Let $\mathbf{T} \in S(\mathbf{y})$ be an acyclic transfer network, so that any flow decomposition of $\mathbf{T}$ is given by a vector $\mathbf{h}$. Pick any such decomposition $\mathbf{h}$. Suppose in addition that, for every $(i,j) \in I_G \times I_R$, there exists at most one path $\rho \in \mathcal{P}_{ij}$ such that $h_\rho > 0$ (if it is the case for one decomposition of $\mathbf{T}$ it has to be the case for all of them). Then we define a *matching* μ of $\mathbf{T}$ as any collection of pairs $(i_1, j_1), \dots, (i_k, j_k)$ such that: $k \geq 2$; $i_\ell \neq i_{\ell'}$ and $j_\ell \neq j_{\ell'}$, for every $\ell \neq \ell'$; $(i_\ell, j_\ell) \in I_G \times I_R$ for every ℓ; and, for every ℓ, there exists a (necessarily unique) path $\rho^{i_\ell j_\ell} \in \mathcal{P}_{i_\ell j_\ell}$ such that $h_{\rho^{i_\ell j_\ell}} > 0$. In this case, we say that the *support* of μ, denoted by supp μ, is the list of pairs involved in the paths $\rho^{i_\ell j_\ell}$, for $\ell = 1, \dots, k$.

Then let $S^{ex}(\mathbf{y})$ be the set of transfer plans in $S(\mathbf{y})$ such that: (i) $\mathbf{T}$ has no cycles; (ii) for any flow decomposition $\mathbf{h}$ of $\mathbf{T}$, and every $(i,j) \in I_G \times I_R$, there exists at most one path $\rho \in \mathcal{P}_{ij}$ such that $h_\rho > 0$; (iii) for every matching $\mu = (i_1, j_1), \cdots, (i_k, j_k)$ of $\mathbf{T}$, either $\mu' = (i_1, j_2), \dots, (i_{k-1}, j_k)(i_k, j_1)$ is not a matching of $\mathbf{T}$, or μ and μ' have the same support.

Then we have the following result.

Proposition A.2 *The set $\{\mathbf{T}^\gamma\}_{\gamma \in \mathcal{C}}$ is the set of directions of $S(\mathbf{y})$, and $S^{ex}(\mathbf{y})$ is its set of extreme points. In particular, $S^{ex}(\mathbf{y})$ is a finite set $\{\mathbf{T}^1, \cdots, \mathbf{T}^k\}$, and for every matrix $\mathbf{T} \in S(\mathbf{y})$, there exists nonnegative scalars $\lambda_1, \dots, \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = 1$, and nonnegative scalars λ_γ for each $\gamma \in \mathcal{C}$ such that*

$$\mathbf{T} = \sum_{\ell=1}^{k} \lambda_\ell \mathbf{T}^\ell + \sum_{\gamma \in \mathcal{C}} \lambda_\gamma \mathbf{T}^\gamma.$$

Proof. To see that $\{\mathbf{T}^\gamma\}_{\gamma \in \mathcal{C}}$ is the set of directions of $S(\mathbf{y})$, just note that if $\mathbf{T} \in S(\mathbf{y})$, then, for any $\gamma \in \mathcal{C}$ and any $\lambda > 0$, the transfer plan $\mathbf{T} + \lambda \mathbf{T}^\gamma$ is also in $S(\mathbf{y})$. Furthermore, any flow that is not a cycle, or a combination of cycles, cannot be added to $\mathbf{T}$ without modifying the achieved distribution.

For extreme points, we start by showing that any $\mathbf{T} \in S(\mathbf{y}) \setminus S^{ex}(\mathbf{y})$ can be written as a convex combination of two transfer plans in $S(\mathbf{y})$, and therefore cannot be an extreme point.

First, suppose that $\mathbf{T}$ has a cycle γ, and let $\tau = \min_{(i,j) \in \gamma} t_{ij}$, $\mathbf{T}' = \mathbf{T} - \tau \mathbf{T}^\gamma$ and $\mathbf{T}'' = \mathbf{T} + \tau \mathbf{T}^\gamma$. It is easy to see that $\mathbf{T}'$ and $\mathbf{T}''$ are both in $S(\mathbf{y})$, and that $\mathbf{T} = \frac{1}{2}\mathbf{T}' + \frac{1}{2}\mathbf{T}''$. Hence we can assume that $\mathbf{T}$ is acyclic. Suppose now that there exists a pair $(i,j) \in I_G \times I_R$ with at least two paths ρ and ρ' in $\mathcal{P}_{ij}$ such that $t_{\ell k} > 0$ for every $(\ell, k) \in \rho$ and every

$(\ell, k) \in \rho'$. Then let $\mathbf{T}^{\rho}$ and $\mathbf{T}^{\rho'}$ be the flows respectively defined by $t^{\rho}_{\ell k} = \mathbf{1}_{(\ell,k)\in\rho}$ and $t^{\rho'}_{\ell k} = \mathbf{1}_{(\ell,k)\in\rho'}$, and let $\tau = \min_{(\ell,k)\in\rho} t_{\ell k} > 0$ and $\tau' = \min_{(\ell,k)\in\rho'} t_{\ell k} > 0$. We define the new transfer plans

$$\mathbf{T}_1 = \mathbf{T} - \tau\mathbf{T}^{\rho} + \tau\mathbf{T}^{\rho'},$$

and

$$\mathbf{T}_2 = \mathbf{T} - \tau'\mathbf{T}^{\rho'} + \tau'\mathbf{T}^{\rho}.$$

It is easy to see that they both achieve $\mathbf{y}$ since they are obtained by reassigning to ρ' some of the money that flows from i to j through ρ, or reciprocally. Furthermore, we have

$$\mathbf{T} = \frac{1/\tau}{1/\tau + 1/\tau'}\mathbf{T}_1 + \frac{1/\tau'}{1/\tau + 1/\tau'}\mathbf{T}_2.$$

Now suppose that $\mathbf{T}$ satisfies properties (i) and (ii) but not (iii) in the definition of $S^{ex}(\mathbf{y})$. Let $\mu = (i_1, j_1), \cdots, (i_k, j_k)$ and $\mu' = (i_1, j_2), \ldots, (i_{k-1}, j_k)(i_k, j_1)$ be two matchings of $\mathbf{T}$ with different supports. Then let $\mathbf{T}^{\mu}$ and $\mathbf{T}^{\mu'}$ be the flows defined respectively by $t^{\mu}_{ij} = \mathbf{1}_{(i,j)\in \text{supp } \mu}$, and $t^{\mu'}_{ij} = \mathbf{1}_{(i,j)\in \text{supp } \mu'}$. Because μ and μ' have different supports, we have $\mathbf{T}^{\mu} \neq \mathbf{T}^{\mu'}$. Let $\tau = \min_{(i,j)\in \text{supp } \mu} t_{ij} > 0$ and $\tau' = \min_{(i,j)\in \text{supp } \mu'} t_{ij} > 0$. Consider the new transfer plans

$$\mathbf{T}_1 = \mathbf{T} - \tau\mathbf{T}^{\mu} + \tau\mathbf{T}^{\mu'},$$

and

$$\mathbf{T}_2 = \mathbf{T} - \tau'\mathbf{T}^{\mu'} + \tau'\mathbf{T}^{\mu}.$$

It is easy to see that they both achieve $\mathbf{y}$ since they are only obtained by reassigning to μ' some of the money that flows from sources $i_1, \ldots, i_k$ to the sinks $j_1, \ldots, j_k$ through μ, and in this reassignment, each sink gains τ from one source and loses τ from another, while each source gives an additional τ to one sink, and reduces its transfer to another source by τ (or reciprocally for $\mathbf{T}_2$). Furthermore, we have

$$\mathbf{T} = \frac{1/\tau}{1/\tau + 1/\tau'}\mathbf{T}_1 + \frac{1/\tau'}{1/\tau + 1/\tau'}\mathbf{T}_2.$$

Therefore, all extreme points are in $S^{ex}(\mathbf{y})$. Now, let $\mathbf{T} \in S^{ex}(\mathbf{y})$, and suppose that it is not an extreme point. Because all extreme points are in $S^{ex}(\mathbf{y})$ we can write $\mathbf{T}$ as a convex combination of extreme points $\mathbf{T}^1, \ldots, \mathbf{T}^k$, all in $S^{ex}(y)$ (we do not need cycles because $\mathbf{T}$ is acyclic). Let $\lambda_\ell > 0$ be the weight of each $\mathbf{T}^\ell$ in this decomposition. For each $\ell = 1, \cdots, k$, and each pair $(i, j) \in I_G \times I_R$, let $\rho_\ell^{ij} \in \mathcal{P}_{ij}$ be the unique path between i and j with positive flow in $\mathbf{T}^\ell$. There may be no such path for some ℓ, but if ρ_ℓ^{ij} and $\rho_{\ell'}^{ij}$ exist both, then we must have $\rho_\ell^{ij} = \rho_{\ell'}^{ij}$, for otherwise $\mathbf{T}$ would put a positive flow on both paths which is impossible.

We pick two of these transfer plans $\mathbf{T}^1$ and $\mathbf{T}^2$, and corresponding flow decompositions $\mathbf{h}^1$ and $\mathbf{h}^2$. For every pair $(i, j) \in I_G \times I_R$, let ρ^{ij} be the unique path between i and j with a positive flow from at least one of the two transfer plans. Let $\tau_{ij}^1 = h_{\rho^{ij}}^1$ and $\tau_{ij}^2 = h_{\rho^{ij}}^2$ be the flows of $\mathbf{T}^1$ and $\mathbf{T}^2$ over this path. If a transfer plan has no flow between i and j, we set the corresponding τ_{ij} to 0. We will say that (i, j) is a blue pair if $\tau_{ij}^1 > \tau_{ij}^2$, and a green pair if $\tau_{ij}^1 < \tau_{ij}^2$. Next, consider the following procedure.

First, pick a blue pair (i_1, j_1). There must exist such a pair for otherwise, $\mathbf{T}^1 = \mathbf{T}^2$. Because i_1 is sending more money over the corresponding path in $\mathbf{T}_1$ than in $\mathbf{T}_2$, there must exist an agent $j_2 \in I_R$ to whom i_1 is sending more money through the path $\rho^{i_2 j_2}$ in $\mathbf{T}_2$ than in $\mathbf{T}_1$. That is (i_1, j_2) is a green pair. For obvious reasons $j_1 \neq j_2$. But then, j_2 is receiving more money from i_1 in $\mathbf{T}_2$ than in $\mathbf{T}_1$. Hence, there must exist an agent $i_2 \in I_G$ such that j_2 receives more money from i_2 in $\mathbf{T}_1$ than in $\mathbf{T}_2$. That is (i_2, j_2) is a blue pair. At this point we can build a new green pair (i_2, j_3), but it could be the case that $j_3 = j_1$. If this is the case, we stop the construction, and otherwise we continue in this way. Because there is a finite number of agents, we must end up creating a new pair such that one of the agents involved was already part of a previous pair. The construction stops the first time this happens.

This procedure creates an undirected cycle of alternate blue and green pairs. It may be the case that some pairs in the construction are not part of the cycle, in this case, we keep only the cycle. We relabel the blue pairs in the cycle $(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)$. Then the green pairs are $(i_1, j_2), \cdots, (i_{k-1}, j_k), (i_k, j_1)$. The blue pairs form a matching μ for $\mathbf{T}$.

Indeed, since $\mathbf{T}$ puts a positive weight λ_1 on $\mathbf{T}_1$, and $\mathbf{T}_1$ has a positive flow over each path corresponding to a blue pair, $\mathbf{T}$ must also have a positive flow over these pairs. But the green pairs form a matching μ' for $\mathbf{T}$ as well since $\mathbf{T}_2$ has a positive flow over each path corresponding to a green pair, and $\mathbf{T}$ puts a positive weight on $\mathbf{T}_2$.

If μ and μ' have the same support, then we can drop the pairs involved in the cycle and do the construction above again with the remaining pairs. At some point, we must end up with two matchings μ and μ' with different support, for otherwise $\mathbf{T}_1$ and $\mathbf{T}_2$ would be equal. But then, $\mathbf{T}$ violates condition (iii) in the definition of $S^{ex}(\mathbf{y})$.

The remaining of the proposition is just the classical decomposition of elements of a convex polytope. (See for example Rockafellar, 1972, section 19). ■

We now uncover some properties of the set $S^*(\mathbf{y})$ of solutions to (MCF). We start by introducing the notion of cyclical monotonicity. For intuition, suppose that you are currently transferring one dollar from agent i_1 to agent j_1 at a cost $\hat{c}_{i_1 j_1}$ (hence you are using the shortest path between these two agents), and another dollar from i_2 to j_2 at a cost $\hat{c}_{i_2 j_2}$. The total cost of this redistribution is therefore $\hat{c}_{i_1 j_1} + \hat{c}_{i_2 j_2}$. Another way to achieve the same redistribution, however, would be to transfer one dollar from i_1 to j_2, and one dollar from i_2 to j_1. If it is the case that

$$\hat{c}_{i_1 j_2} + \hat{c}_{i_2 j_1} < \hat{c}_{i_1 j_1} + \hat{c}_{i_2 j_2},$$

then, clearly, the first plan is not optimal.

Definition 1 (Cyclical Monotonicity) *We say that a subset $\Gamma \subseteq I_G \times I_R$ is $\hat{\mathbf{c}}$-cyclically monotone if for every sequence $(i_1, j_1), \cdots, (i_k, j_k)$ of points in Γ such that all sources and all sinks are distinct, we have*

$$\sum_{\ell=1}^{k} \hat{c}_{i_\ell j_\ell} \leq \sum_{\ell=1}^{k} \hat{c}_{i_\ell j_{\ell+1}}$$

with the convention $j_{k+1} = j_1$.

Now, let Γ be the subset of $I_G \times I_R$ such that a pair (i, j) belongs to Γ if there exists an optimal transfer plan $T \in S^*(\mathbf{y})$, a flow decomposition $\mathbf{h}$ of $\mathbf{T}$ (we know that $\mathbf{T}$ is acyclic), and a shortest path ρ from i to j such that $h_\rho > 0$ (we know that all positive path flows of $\mathbf{T}$ are on shortest paths).

Proposition A.3 Γ *is* $\hat{\mathbf{c}}$*-cyclically monotone.*

Proof. Suppose otherwise, and let $(i_1, j_1), \cdots, (i_k, j_k)$ be a collection of points over which the monotonicity condition fails. For each of the pairs $\ell = 1, \cdots, k$, let $\mathbf{T}^\ell$ be an optimal transfer plan with flow decomposition $\mathbf{h}^\ell$ that is positive on a shortest path from i_ℓ to j_ℓ. Then $\mathbf{T} = \frac{1}{k}\mathbf{T}^1 + \cdots + \frac{1}{k}\mathbf{T}^k$ is also an optimal transfer plan whose flow decomposition $\mathbf{h} = \frac{1}{k}\sum_{\ell=1}^k \mathbf{h}^\ell$ has a positive flow between all these pairs. Let $\underline{h}$ be the minimum flow intensity across all pairs. Now consider reassigning $\underline{h}$ from each pair (i_ℓ, j_ℓ) to the pair $(i_\ell, j_{\ell+1})$. This reassignment does not change the final distribution and it leads to a cost reduction of

$$\underline{h} \times \left(\sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}} - \sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} \right) > 0,$$

a contradiction to the optimality of $\mathbf{T}$. ■

A natural corollary of this result is the following.

Corollary A.1 *Let* $(i_1, j_1), \cdots, (i_k, j_k)$ *be a sequence of points in* Γ*, and suppose that, for every* $\ell = 1, \ldots, k$*,* $(i_\ell, j_{\ell+1})$ *is also in* Γ *(with the convention* $j_{k+1} = j_1$*). Then*

$$\sum_{\ell=1}^k \hat{c}_{i_\ell j_\ell} = \sum_{\ell=1}^k \hat{c}_{i_\ell j_{\ell+1}}$$

Proof. By the cyclical monotonicity inequality, the left-hand side is smaller than the right-hand side. But by rearranging the order of the pairs, we can also obtain the reverse inequality as a cyclical monotonicity inequality. ■

Equipped with this set of results, we can state a sufficient condition for the uniqueness of the cost minimizing transfer plan.

Proposition A.4 (Uniqueness) *Suppose that the cost vector* $\mathbf{c}$ *and the target distribution* $\mathbf{y}$ *satisfy the following properties: (a) for every* $(i, j) \in I_G \times I_R$*, there is a unique* *shortest path in $\mathcal{P}_{ij}$; and (b) for every sequence $(i_1, j_1), \cdots, (i_k, j_k)$ of points in Γ such that all sources and all sinks are distinct, we have either $\sum_{\ell=1}^{k} \hat{c}_{i_\ell j_\ell} < \sum_{\ell=1}^{k} \hat{c}_{i_\ell j_{\ell+1}}$, or the list of arcs in the shortest paths $\rho^{i_1, j_1}, \cdots, \rho^{i_k, j_k}$ and the list of arcs in the shortest paths $\rho^{i_1, j_2}, \cdots, \rho^{i_k, j_1}$ are the same. Then $S^*(\mathbf{y})$ is a singleton.*

Proof. Suppose that $S^*(\mathbf{y})$ is not a singleton. Then there exists a transfer plan $\mathbf{T}$ in $S^*(\mathbf{y})$ that is not in $S^{ex}(\mathbf{y})$. Since $\mathbf{T}$ must be acyclic, it must fail property (ii) or (iii) of the definition of $S^{ex}(\mathbf{y})$. Suppose first that it fails property (ii). Then there must exist a pair $(i, j) \in I_G \times I_R$ and a flow decomposition $\mathbf{h}$ of $\mathbf{T}$ with positive flows on two distinct paths $\rho \neq \rho'$ of $\mathcal{P}_{ij}$. By Theorem A.1, both of these paths must be shortest paths, but that contradicts (a). Suppose now that it satisfies property (ii) but fails (iii). Then let $\mu = (i_1, j_1), \cdots, (i_k, j_k)$ and $\mu' = (i_1, j_2), \cdots, (i_k, j_1)$ be two matchings of $\mathbf{T}$ with different support such that a flow decomposition $\mathbf{h}$ of $\mathbf{T}$ puts positive flows on μ and μ'. Then, by Corollary A.1, we must have $\sum_{\ell=1}^{k} \hat{c}_{i_\ell j_\ell} = \sum_{\ell=1}^{k} \hat{c}_{i_\ell j_{\ell+1}}$, which contradicts (b). ■

This allows us to prove the following generic uniqueness result.

Proposition A.5 (Generic Uniqueness) *Generically in $\mathbf{c}$, the cost minimizing transfer plan is unique for every $\mathbf{y}^0$ and every feasible $\mathbf{y}$.*

Proof. Consider the set of cost vectors $\tilde{C}$ such that: (i) for every pair of agents (i, j), and every pair of distinct paths $\rho \neq \rho'$ in $\mathcal{P}_{ij}$, $c_\rho \neq c_{\rho'}$; and (ii), for every sequence $(i_1, j_1), \cdots, (i_k, j_k)$ of arcs in A, such that $(i_\ell, j_{\ell+1}) \in A$ for each ℓ, and every choice of paths $\rho_\ell \in \mathcal{P}_{i_\ell j_\ell}$, and $\rho'_\ell \in \mathcal{P}_{i_\ell j_{\ell+1}}$, such that the list of arcs in he paths $\left(\rho\right)_{\ell=1,\ldots,k}$ and $\left(\rho'\right)_{\ell=1,\ldots,k}$ are distinct, then $\sum_{\ell=1}^{k} c_{\rho_\ell} \neq \sum_{\ell=1}^{k} c_{\rho'_\ell}$.

Note that we need to assume that the list or arcs in ρ and ρ' are distinct, for otherwise the two sums are necessarily equal. It is easy that any cost vector in $\tilde{C}$ satisfies properties (a) and (b) of proposition A.4 for any $\mathbf{y} \in Y$. But the set of cost vectors $\mathbf{c}$ that do not belong to $\tilde{C}$ is a finite reunion of hyperplanes defined by a linear inequality, therefore $\tilde{C}$ is generic in the set of possible cost vectors. ■

Before going back to the original problem, we provide some results on the value function of the cost minimization problem. As usual in linear programming, the minimization program has a dual maximization program. In this case, it can be written as a program over a vector of "prices" $\boldsymbol{\phi}$ in $\mathbb{R}^N$. For any feasible $\mathbf{y}$,

$$c\left(\mathbf{y}, \mathbf{y}^0\right) = \max_{\boldsymbol{\phi}} \sum_{i=1}^{N} \phi_i (y_i - y_i^0) \qquad \text{s.t. } \phi_j - \phi_i \leq c_{ij} \;\; \forall i \neq j.$$

As a consequence, we have the following result.

Proposition A.6 *The cost function $c\left(\mathbf{y}, \mathbf{y}^0\right)$ is convex in $\mathbf{y}$ and $\mathbf{y}^0$, supermodular in $\mathbf{y} - \mathbf{y}^0$, concave in $\mathbf{c}$ and continuous in all variables. Furthermore, it depends on $\mathbf{c}$ only through $\hat{\mathbf{c}}$. $S^*(\mathbf{y})$ is upper hemicontinuous in $\mathbf{y}$, $\mathbf{y}^0$ and $\mathbf{c}$.*

Proof. In the original problem, we are minimizing an objective function that is linear in $\hat{\mathbf{c}}$ over a convex set. In the dual problem, we are maximizing an objective function that is linear in $\mathbf{y}$ and $\mathbf{y}^0$ over a convex set. The dual formulation of the problem maximizes a supermodular function in $(\mathbf{y} - \mathbf{y}^0, \boldsymbol{\phi})$ over a lattice, therefore its value function is supermodular. The continuity properties can be derived by applying the maximum theorem (see, for example, Aliprantis and Border, 2006) to the minimization problem after having reduced the space over which the function is minimized to the compact set of acyclic transfers that achieve $\mathbf{y}$. The fact that the value function depends only on $\hat{\mathbf{c}}$ is a direct consequence of Theorem A.1. ■

Going back to the original problem, we can now rewrite the problem of maximizing the potential, as $\max_{\mathbf{y} \in Y} \; \sum_{i=1}^{n} U_i(y_i) - c\left(\mathbf{y}, \mathbf{y}^0\right)$, where $U_i(y_i) = \int_1^{y_i} \ln(u_i'(x))dx$, and summarize our results in the following theorem.

Theorem A.2 *There is a unique equilibrium distribution $\mathbf{y}^*$. It is continuous as a function of $\mathbf{y}^0$ and $\mathbf{c}$, and depends on $\mathbf{c}$ only through $\hat{\mathbf{c}}$. The set of Nash equilibria of the transfer game is a nonempty, compact and convex polytope given by*

$$S^* = \arg\min_{\mathbf{T}} \sum_{1 \leq i,j \leq N} c_{ij} t_{ij} \qquad s.t. \; \sum_{i \neq j} (t_{ji} - t_{ij}) = y_i^* - y_i^0 \quad \forall i.$$

It is generically a singleton. Furthermore, every transfer network in S^* is acyclic, and all *its positive flows are on shortest paths. As a correspondence, S^* is upper hemicontinuous in $(\mathbf{y}^0, \mathbf{c})$, and depends on $\mathbf{c}$ only through $\hat{\mathbf{c}}$.*

Proof. Most points are direct consequences of our results on the cost minimization problem.The only point that needs proof is the existence, uniqueness and continuity of the solution to the first program. Note first that Y is closed, and bounded since no agent can get more than $\sum_i y_i^0$. The objective function is continuous, by assumption for the first term, and as a consequence of convexity for the cost term. That gives us existence. The program is strictly concave in $\mathbf{y}$, by strict concavity of the $U_i(\cdot)$ functions, and convexity of $c(\cdot, \mathbf{y}^0)$, therefore the solution is unique. Furthermore, the maximum theorem implies that the solution to the problem is continuous in $\mathbf{y}^0$ and $\mathbf{c}$. ■

APPENDIX B: Best-response dynamics

In this section, we use the potential to show convergence of best-response dynamics. As a preliminary, we show a few lemmas.

Lemma B.1 *For every scalar λ, the set $\Phi_\lambda = \{\mathbf{T} : \varphi(\mathbf{T}) \geq \lambda\}$ is compact.*

Proof. We know that $\varphi(\cdot)$ attains its maximum over the set of transfers. Let $\overline{\varphi}$ denote the value of this maximum. The set Φ_λ is the reciprocal image of the interval $[\lambda, \overline{\varphi}]$ by the continuous function $\varphi(\cdot)$, hence it is closed (if $\lambda > \overline{\varphi}$, then Φ_λ is empty, and the lemma holds vacuously). Suppose, by contradiction, that it is unbounded, and let $\{\mathbf{T}^n\}$ be an unbounded sequence of transfers in Φ_λ, so that $\|\mathbf{T}^n\| \to \infty$. Fix a scalar $K \geq 0$. We can assume that, for every n, $\|\mathbf{T}^* - \mathbf{T}^n\| \geq K$. Let $\mathbf{T}^*$ be a maximizer of $\varphi(\cdot)$. Clearly, $T^* \in \Phi_\lambda$. Consider the sequence $\tilde{\mathbf{T}}^n$ defined by

$$\tilde{\mathbf{T}}^n = \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\mathbf{T}^n + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\mathbf{T}^*$$

Note that, for every n, $\left\|\tilde{\mathbf{T}}^n - \mathbf{T}^*\right\| = K$. By concavity of φ, we have

$$\begin{aligned}\overline{\varphi} \geq \varphi(\tilde{\mathbf{T}}^n) &\geq \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\varphi(\mathbf{T}^n) + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\varphi(\mathbf{T}^*) \\ &\geq \frac{K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\lambda + \frac{\|\mathbf{T}^* - \mathbf{T}^n\| - K}{\|\mathbf{T}^* - \mathbf{T}^n\|}\overline{\varphi} \xrightarrow{n\to\infty} \overline{\varphi}\end{aligned}$$

Hence the sequence $\varphi(\tilde{\mathbf{T}}^n)$ converges to $\overline{\varphi}$. Since the sequence $\tilde{\mathbf{T}}^n$ lies in the compact set of points at distance K of $\mathbf{T}^*$, it has a converging subsequence. Let $\tilde{\mathbf{T}}^\infty$ denote the limit of this subsequence. It is at distance K of $\mathbf{T}^*$. By continuity of φ, we must have $\varphi(\tilde{\mathbf{T}}^\infty) = \overline{\varphi}$. Therefore, we have found a maximizer of φ at distance K of $\mathbf{T}^*$. Since we can do so for every K, this implies that the set S^* is unbounded, a contradiction since it is compact by theorem A.2. ■

For a player i, and a transfer profile $\mathbf{T}$, let $BR_i(\mathbf{T})$ be the set of pairs $(\mathbf{T}'_i, \mathbf{T}_{-i})$ such that $\mathbf{T}'_i$ is a best response to $\mathbf{T}_{-i}$. For any ordering (a permutation) σ of players let

$$BR^\sigma(\mathbf{T}) = BR_{\sigma(n)} \circ BR_{\sigma(n-1)} \circ \cdots \circ BR_{\sigma(1)}(\mathbf{T}).$$

Lemma B.2 *For any ordering σ, the best response correspondence BR^σ is nonempty, compact-valued and upper hemicontinuous. Furthermore, the set of fixed points of BR^σ is exactly the set S^* of Nash equilibria of the transfer game.*

Proof. First consider $BR_i(\cdot)$. For every $\mathbf{T}' \in BR_i(\mathbf{T})$, we have $\mathbf{T}'_{-i} = \mathbf{T}_{-i}$, hence the correspondence is continuous in all dimensions $j \neq i$. On dimension i, we have, by the best response potential property,

$$\mathbf{T}'_i \in \arg \max_{\hat{\mathbf{T}}_i \in \mathbb{R}_+^{n-1}} \varphi(\hat{\mathbf{T}}_i, \mathbf{T}_{-i})$$

When solving this program, we can fix a transfer plan $\mathbf{T}_i^0$ for player i, and restrict the program above to transfers in the set

$$\mathcal{S}(\mathbf{T}_{-i}) = \left\{\hat{\mathbf{T}}_i \,:\, (\hat{\mathbf{T}}_i, \mathbf{T}_{-i}) \in \Phi_{\varphi(\mathbf{T}_i^0, \mathbf{T}_{-i})}\right\},$$

which is compact by lemma B.1. It is also easy to see that the correspondence $\mathcal{S}(\cdot)$ is continuous. By continuity of $\varphi(\cdot)$, we can apply the maximum theorem to conclude that the maximizer correspondence of this program is nonempty, compact-valued and upper hemicontinuous. This implies that $BR_i(\cdot)$ is nonempty, compact-valued and upper hemicontinuous, for every i, and therefore that $BR^\sigma(\cdot)$ satisfies these properties as well.

If $\mathbf{T}$ is a Nash equilibrium of the transfer game, it is clearly a fixed point of BR^σ. Suppose that $\mathbf{T} \in BR^\sigma(\mathbf{T})$, then φ remains constant along the sequence of best replies that lead to $BR^\sigma(\mathbf{T})$. This implies that, at each step i of this sequence, $\mathbf{T}$ is among the best replies of player i. Therefore T is a Nash equilibrium of the transfer game. ■

We say that $\{\mathbf{T}^k\}$ is a best-response dynamics sequence if, for every k,

$$\mathbf{T}^k \in \underbrace{BR^\sigma \circ \cdots \circ BR^\sigma}_{k \text{ times}}(\mathbf{T}^0).$$

Proposition B.1 *The limit set of any best-response dynamics sequence is a subset of the set of Nash equilibria S^*. For any best-response dynamics sequence $\{\mathbf{T}^k\}$, the sequence of corresponding consumption profiles $\{\mathbf{y}^k\}$ converges to the unique equilibrium distribution $\mathbf{y}$.*

Proof. Pick any best-response dynamics sequence $\{\mathbf{T}^k\}$. First note that the sequence $\varphi(\mathbf{T}^k)$ is increasing and bounded above by $\overline{\varphi}$, and therefore converges. We denote its limit by φ_∞. Since, for every k, $\varphi(\mathbf{T}^k) \geq \varphi(\mathbf{T}^0)$, the sequence $\mathbf{T}^k$ lies in the compact set $\Phi_{\varphi(\mathbf{T}^0)}$. Hence $\{\mathbf{T}^k\}$ admits a converging subsequence. Let $\{T^{g(k)}\}$ be such a subsequence, and $\mathbf{T}^{g(\infty)}$ its limit. By continuity of φ, we have $\varphi\big(\mathbf{T}^{g(\infty)}\big) = \varphi_\infty$.

Consider the subsequence $\mathbf{T}^{g(k)+1}$. Since it lies in a compact set, we can extract a converging subsequence from this new sequence. Assume, without loss of generality for the argument to follow, that $\mathbf{T}^{g(k)+1}$ is itself convergent, and denote its limit by $\tilde{\mathbf{T}}$. Since $\mathbf{T}^{g(k)+1} \in BR^\sigma(\mathbf{T}^{g(k)})$, we have by upper hemicontinuity of BR^σ, that $\tilde{\mathbf{T}} \in BR^\sigma(\mathbf{T}^{g(\infty)})$. Since $\mathbf{T}^{g(k)+1}$ is also a converging subsequence of $\mathbf{T}^k$, we have $\varphi(\tilde{\mathbf{T}}) = \varphi_\infty = \varphi(\mathbf{T}^{g(\infty)})$. This implies that $\mathbf{T}^{g(\infty)}$ is also in $BR^\sigma(\mathbf{T}^{g(\infty)})$, and is therefore in S^*. This must hold for any limit of a converging subsequence of $\mathbf{T}^k$, so the limit set of $\mathbf{T}^k$ is a subset of S^*.

Next consider the sequence of consumption profiles $\mathbf{y}^k$. Since $\{\mathbf{T}^k\}$ lies in the compact set $\Phi_{\varphi(\mathbf{T}^0)}$, and since the function that maps a transfer profile to the corresponding distribution is continuous, it is uniformly continuous on $\Phi_{\varphi(\mathbf{T}^0)}$. Pick $\varepsilon > 0$, and $\delta > 0$, such that for every $\mathbf{T}$ and $\mathbf{T}'$ in $\Phi_{\varphi(\mathbf{T}^0)}$, $\|\mathbf{y} - \mathbf{y}'\| < \delta$. By definition of the limit set, we can pick K, such that for every $k > K$, $\mathbf{T}^k$ is within distance δ of the limit set S^* of the sequence. Then, for every $k > K$, $\mathbf{y}^k$ is within distance ε of the distribution $\mathbf{y}$ associated with some $\mathbf{T} \in S^*$, that is, $\left\|\mathbf{y}^k - \mathbf{y}\right\| < \varepsilon$. Hence $\mathbf{y}^k$ converges to the unique equilibrium distribution $\mathbf{y}$. ■

Clearly, if the equilibrium transfer network is unique, as is generically true, then best-response dynamics converges to the unique equilibrium.

APPENDIX C : Transfer Intermediaries

In this section, we prove Theorem 2 of the paper, and then we provide an example showing how transitivity may be satisfied by an altruistic network that is inconsistent with deferential caring.

Proof of Theorem 2. For the first point, suppose first that $\boldsymbol{\alpha}$ is transitive and consider an equilibrium $\mathbf{T}$ with a transfer chain $t_{i_1 i_2} > 0$, ..., $t_{i_{\ell-1} i_\ell} > 0$ with $\ell \geq 3$. Let τ be the smallest transfer in the chain. Consider the alternative profile $\mathbf{T}'$ where $t'_{i_s i_{s+1}} = t_{i_s i_{s+1}} - \tau$, $t'_{i_1 i_l} = t_{i_1 i_l} + \tau$ and $t'_{ij} = t_{ij}$ for the other pairs. In $\mathbf{T}'$, transfer τ is redirected to flow directly from i_1 to i_ℓ rather than indirectly through the chain. This removes one link in the transfer chain. Note that consumption is unchanged. In addition, the difference in costs between the original and the modified profiles is equal to $c_{i_1 i_\ell} - \sum_{s=1}^{\ell-1} c_{i_s i_{s+1}}$. By Theorem 1, i_1, ..., i_ℓ is a least path of $\boldsymbol{\alpha}$, which means that $\hat{c}_{i_1 i_\ell} = \sum_{s=1}^{\ell-1} c_{i_s i_{s+1}}$. Since $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$, $\mathbf{T}'$ has the same cost as $\mathbf{T}$. By Theorem 1, this is also a Nash equilibrium. Thus, for any Nash equilibrium with a transfer chain, we can construct another equilibrium with one less link in the transfer chain. Repeating the operation eventually leads to an equilibrium without transfer chains.

Next, suppose that $\boldsymbol{\alpha}$ is not transitive. Then, there exists some pair i, j such that $\alpha_{ij} < \hat{\alpha}_{ij}$. Set $y_i^0 = Y$ and $y_k^0 = 0 \; \forall k \neq i$. By Theorem 3, y_i is increasing in Y. Suppose that y_i is bounded. Then, y_i tends to some y. Since $\sum_j y_j = Y$, there is some k such that y_k tends to ∞. Here all the money originates in i, so money must flow somehow from i to k. Thus, $u_i'(y_i) = \hat{\alpha}_{ik} u_k'(y_k)$. In the limit, this yields: $u_i'(y) = 0$ which is a contradiction. Therefore, y_i becomes arbitrarily large as Y increases. From conditions (5) in the main paper, we know that $u_i'(y_i) \geq \hat{\alpha}_{ij} u_j'(y_j)$. Since y_i tends to ∞ and $\hat{\alpha}_{ij} > 0$, $u_j'(y_j)$ tends to 0 and hence $y_j > 0$ if Y is large enough. Money flows, somehow, from i to j. By Theorem 1, it flows through a least cost path which, by assumption, cannot be the direct link.

For the second point, suppose that $\boldsymbol{\alpha}$ is consistent with deferential caring and let $\mathbf{B}$ be the matrix of weights that agents put on others social utilities. Let $\mathbf{M} = (\mathbf{I} - \mathbf{B})^{-1}$ such that $\alpha_{ij} = m_{ij}/m_{ii}$ as in Section II. We can easily show that $\boldsymbol{\alpha}$ is transitive iff $\forall i, j, k, \alpha_{ik} \geq \alpha_{ij}\alpha_{jk}$. This is equivalent to: $\forall i, j, k, m_{ik}m_{jj} \geq m_{ij}m_{jk}$. These inequalities are called the "path product conditions", and are known to hold if $\mathbf{M}$ is the inverse of a M-matrix, see Johnson & Smith (2007). This is the case here. ■

Next, we provide an example of a transitive altruistic network that is inconsistent with deferential caring. For this, we adapt the example of Johnson & Smith (2011, p. 963). Consider the following altruistic network connecting 4 agents

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0.1 & 0.4 & 0.3 \\ 0.4 & 0 & 0.4 & 0.65 \\ 0.1 & 0.2 & 0 & 0.6 \\ 0.15 & 0.3 & 0.6 & 0 \end{pmatrix}$$

which is transitive since $\forall i, j, k$ distinct, $\alpha_{ij} \geq \alpha_{ik}\alpha_{kj}$. Suppose that $\boldsymbol{\alpha}$ is consistent with deferential caring. Then there exists $\mathbf{B} \geq \mathbf{0}$ such that $b_{ii} = 0$, $\lambda_{\max}(\mathbf{B}) < 1$ and $\alpha_{ij} = m_{ij}/m_{ii}$ with $\mathbf{M} = (\mathbf{I} - \mathbf{B})^{-1}$. Let $\mathbf{D}$ be the diagonal matrix such that $d_{ii} = 1/m_{ii}$. Then, $\mathbf{I} + \boldsymbol{\alpha} = \mathbf{D}(\mathbf{I} - \mathbf{B})^{-1} \Rightarrow \mathbf{B} = \mathbf{I} - (\mathbf{I} + \boldsymbol{\alpha})^{-1}\mathbf{D}$. Since $b_{ii} = 0$, we must have $d_{ii} = 1/[(\mathbf{I} + \boldsymbol{\alpha})^{-1}]_{ii}$. This implies that

$$\mathbf{B} \approx \begin{pmatrix} 0 & 0.003 & 0.231 & 0.054 \\ 0.376 & 0 & -0.074 & 0.425 \\ 0.004 & 0.031 & 0 & 0.510 \\ 0.035 & 0.281 & 0.588 & 0 \end{pmatrix}$$

which is impossible since $b_{23} < 0$. This provides an example of transitive altruistic network that is not consistent with deferential caring. By contrast, Theorem 3.2 of Johnson & Smith (1999, p. 183) implies that for $n = 2$ or 3, any transitive altruistic network is consistent with deferential caring.

APPENDIX D : Comparative Statics

We start this section by proving a technical lemma showing that, generically, the inequalities in equilibrium conditions (5) of the paper hold strictly. This result is important for the proofs of the comparative statics results in the paper. Then we extend example 3 of the main paper, by showing that if society consists of two separate communities with distinct aggregate incomes, an inequality-reducing income redistribution from rich individuals in the poorest to poor individuals in the richest community increases consumption inequality whenever the income gap between the two communities is sufficiently large. Finally, we conclude the section by illustrating the comparative statics result of Theorem 4 with an example that shows the evolution of the transfer network and equilibrium consumption as altruism increases between two agents in the network. In particular, we exhibit non-monotonic consumption changes for some agents.

(a) Genericity Result.

First, we show the genericity result in the sense of measure, which is the one adapted in the paper.

Lemma D.1 *Generically in* $(\boldsymbol{\alpha}, \mathbf{y}^0)$*, the unique equilibrium transfer network* $\mathbf{T}$ *satisfies* $t_{ij} = 0 \Rightarrow u_i'(y_i) > \alpha_{ij} u_j'(y_j)$.

Proof. Consider the set $\mathcal{A}$ of altruistic networks with no zeros (that is $\alpha_{ij} > 0$ for all $i \neq j$), and the set of initial income distributions $\mathcal{Y}$. Note that the set of altruistic networks with some zeros has measure 0, but our proof would work if we restricted ourselves to a set of altruistic networks with zeros on some given arcs. Let G be the set of oriented acyclic graphs whose vertices are the agents of our model. For a graph $\mathbf{g} \in G$, we let $g_{ij} = 1$ if (i, j) is an arc of $\mathbf{g}$,and $g_{ij} = 0$ otherwise. For every pair (i, j), let G_{ij} be the set of graphs in G such that i and j are not path-connected.

Next, we pick a pair (i, j) and a graph $\mathbf{g} \in G_{ij}$. Denote by $C_i^{\mathbf{g}}$ and $C_j^{\mathbf{g}}$ the connected components of i and j. For any $k \in C_i^{\mathbf{g}}$, there is a unique undirected path connecting i to k. For every arc (ℓ, m) on this path, let $\beta_{\ell m} = \alpha_{\ell m}^{g_{m\ell} - g_{\ell m}}$, and let β_{ik} be the product of the $\beta_{\ell m}$ along this path. Then we can define the functions

$$h_k^{\mathbf{g}}(x) = \left[u_k'\right]^{-1}\left(\beta_{ik} u_i'(x)\right),$$

and $h_i^{\mathbf{g}}(x) = x$. Note that these functions are strictly increasing in x and continuous in x and $\boldsymbol{\alpha}$. Then the sum $\sum_{k \in C_i^{\mathbf{g}}} h_k^{\mathbf{g}}(x)$ is also a strictly increasing and continuous in x, and continuous in $\boldsymbol{\alpha}$, and so is its inverse which we denote by $H_i^{\mathbf{g}}(x)$. We define similarly the strictly increasing and continuous function $H_j^{\mathbf{g}}(x)$ for j.

Now consider the set $\mathcal{E}_{ij}^{\mathbf{g}}$ of initial income profiles $\mathbf{y}^0$ and altruistic networks $\boldsymbol{\alpha}$ that satisfy $H_i^{\mathbf{g}}\left(y^0(C_i^{\mathbf{g}})\right) = H_j^{\mathbf{g}}\left(y^0(C_j^{\mathbf{g}})\right)$. Because the two functions $H_i^{\mathbf{g}}(\cdot)$ and $H_j^{\mathbf{g}}(\cdot)$ are strictly increasing, one can write y_i^0 as a continuous function of $(\mathbf{y}_{-i}^0, \boldsymbol{\alpha})$. Therefore the set $\mathcal{E}_{ij}^{\mathbf{g}}$ has Lebesgue measure 0 in $\mathcal{A} \times \mathcal{Y}$ as the graph of a continuous function (see, for example, Zorich & Cooke, 2004). But then the set

$$\mathcal{E} = \bigcup_{(i,j)} \bigcup_{\mathbf{g} \in G_{ij}} \mathcal{E}_{ij}^{\mathbf{g}},$$

also has measure 0 in the set of initial income profiles, as a finite union of measure 0 sets.

Note that the set of $(\boldsymbol{\alpha}, \mathbf{y}^0)$ such that $\boldsymbol{\alpha}$ has some zeros has measure 0, and that the set of $(\boldsymbol{\alpha}, \mathbf{y}^0)$ such that $\boldsymbol{\alpha}$ does not satisfy the generic uniqueness conditions also has measure 0. To conclude the proof, we show that the set of remaining $(\boldsymbol{\alpha}, \mathbf{y}^0)$ that do not satisfy the property of the lemma is a subset of $\mathcal{E}$, and therefore has measure 0. To see that, suppose that in the unique equilibrium, there exists a pair (i,j) such that $t_{ij} = 0$ and $u_i'(y_i) = \alpha_{ij} u_j'(y_j)$. First, note that i and j cannot be connected in the equilibrium transfer network $\mathbf{T}$. Otherwise one could transfer a sufficiently small amount ε over the arc (i,j), substract ε from all transfers along the undirected path that connects i to j and go in the opposite direction as (i,j) and add ε to all such transfers that go in the same direction as (i,j), and still satisfy the equilibrium conditions. This would violate equilibrium uniqueness. But then if we let $\mathbf{g}$ be the graph of the unique equilibrium transfer network, we have $\mathbf{g} \in G_{ij}$. And equilibrium conditions (5) from the paper imply that $(\boldsymbol{\alpha}, \mathbf{y}^0)$ must be in $\mathcal{E}_{ij}^{\mathbf{g}}$. ■

Note that it is also possible to prove genericity in a topological sense: the set of $(\boldsymbol{\alpha}, \mathbf{y}^0)$ such that $t_{ij} = 0 \Rightarrow u_i'(y_i) > \alpha_{ij} u_j'(y_j)$ is open and dense in the product set of altruistic networks and initial income profiles.

(b) Inequality increasing redistribution.

Consider an altruistic network formed of two communities C_1 and C_2. Communities are separate but strongly connected within. Formally, $\forall i \in C_1, j \in C_2, \alpha_{ij} = \alpha_{ji} = 0$ and $\forall i, j \in C_1$ (or C_2), $\hat{\alpha}_{ij} > 0$. Assume that $y^0(C_2) > y^0(C_1)$ so that C_2 is richer, overall, than C_1. We can show the following result.

Proposition D.1 *Consider an income inequality reducing redistribution from C_1 to C_2. For any value of $y^0(C_1)$, there exists Y_2 such that if $y^0(C_2) \geq Y_2$, consumption inequality increases in terms of second-order stochastic dominance.*

Proof. To prove this result, we first bound each agent's consumption by functions of aggregate income. Consider community C_2. From the equilibrium conditions, we have: $\forall i, j, u_i'(y_i) \leq \hat{\alpha}_{ij} u_j'(y_j) \Rightarrow (u_j')^{-1}(\frac{1}{\hat{\alpha}_{ij}} u_i'(y_i)) \geq y_j$. Let $f_i(y_i) = \sum_j (u_j')^{-1}(\frac{1}{\hat{\alpha}_{ij}} u_i'(y_i))$. Summing over j yields $f_i(y_i) \geq \sum_j y_j = \sum_j y_j^0 = y^0(C_2)$. In addition, f_i is increasing. As y_i tends to ∞, $\frac{1}{\hat{\alpha}_{ij}} u_i'(y_i)$ tends to 0 and hence $(u_j')^{-1}(\frac{1}{\hat{\alpha}_{ij}} u_i'(y_i))$ tends to ∞. Therefore, $y_i \geq f_i^{-1}(y^0(C_2))$ where f_i^{-1} is increasing and satisfies $\lim_{y \to \infty} f_i^{-1}(y) = \infty$. This implies that consumption of every agent becomes arbitrarily large as the aggregate community income becomes arbitrarily large.

Conversely, consider community C_1. $\forall j, i, u_j'(y_j) \leq \hat{\alpha}_{ji} u_i'(y_i) \Rightarrow y_j \geq (u_j')^{-1}(\hat{\alpha}_{ji} u_i'(y_i))$. Let $g_i(y_i) = \sum_j (u_j')^{-1}(\hat{\alpha}_{ji} u_i'(y_i))$ and sum over j. We obtain: $y^0(C_1) \geq g_i(y_i)$ where g_i satisfies similar properties as f_i. This implies that $y_i \leq g_i^{-1}(y^0(C_1))$ where g_i^{-1} is increasing.

By Theorem 3, the redistribution decreases weakly the consumption of every agent in C_1 and increases weakly the consumption of every agent in C_2. This increases inequality for second-order stochastic dominance if the initial income profile satisfies $\max_{i \in C_1} y_i \leq \min_{i \in C_2} y_i$. In other words, if the richest agent in terms of consumption in the poor community is poorer than the poorest agent in the rich community. This is satisfied if $\max_{i \in C_1} g_i^{-1}(y^0(C_1)) \leq \min_{i C_2} f_i^{-1}(y^0(C_2))$. The fact that $\lim_{y \to \infty} f_i^{-1}(y) = \infty$ then proves the result. ∎

(c) Increasing altruism, an example.

In this section, we illustrate global comparative statics with respect to altruism levels by an example. In the example, all agents have identical CARA utilities $u_i(y_i) = -e^{-y_i}$. The altruistic network is given in Figure 1. We vary the altruism level $\alpha_{36} = e^{-c}$. As shown in Theorem 4 of the paper, the transfer network is locally stable for generic values of c, and agents who are connected to 3 consume less, while agents connected to 6 consume more. Globally, agents may be connected to 3 at some point, and 6 at another (agents 2 and 5 in the example), and their consumption is non-monotonic. Note that the connected components of both agents 3 and 6 shrink and expand at times as we vary c. The values of c for which the graph of the transfer network changes correspond to non-generic altruistic networks at which there are multiple equilibrium transfer networks. For example, at the transition between transfer graphs A and B, at $c = 24.5$, there are multiple equilibria which correspond to the convex combinations of the left and right limit transfer networks.

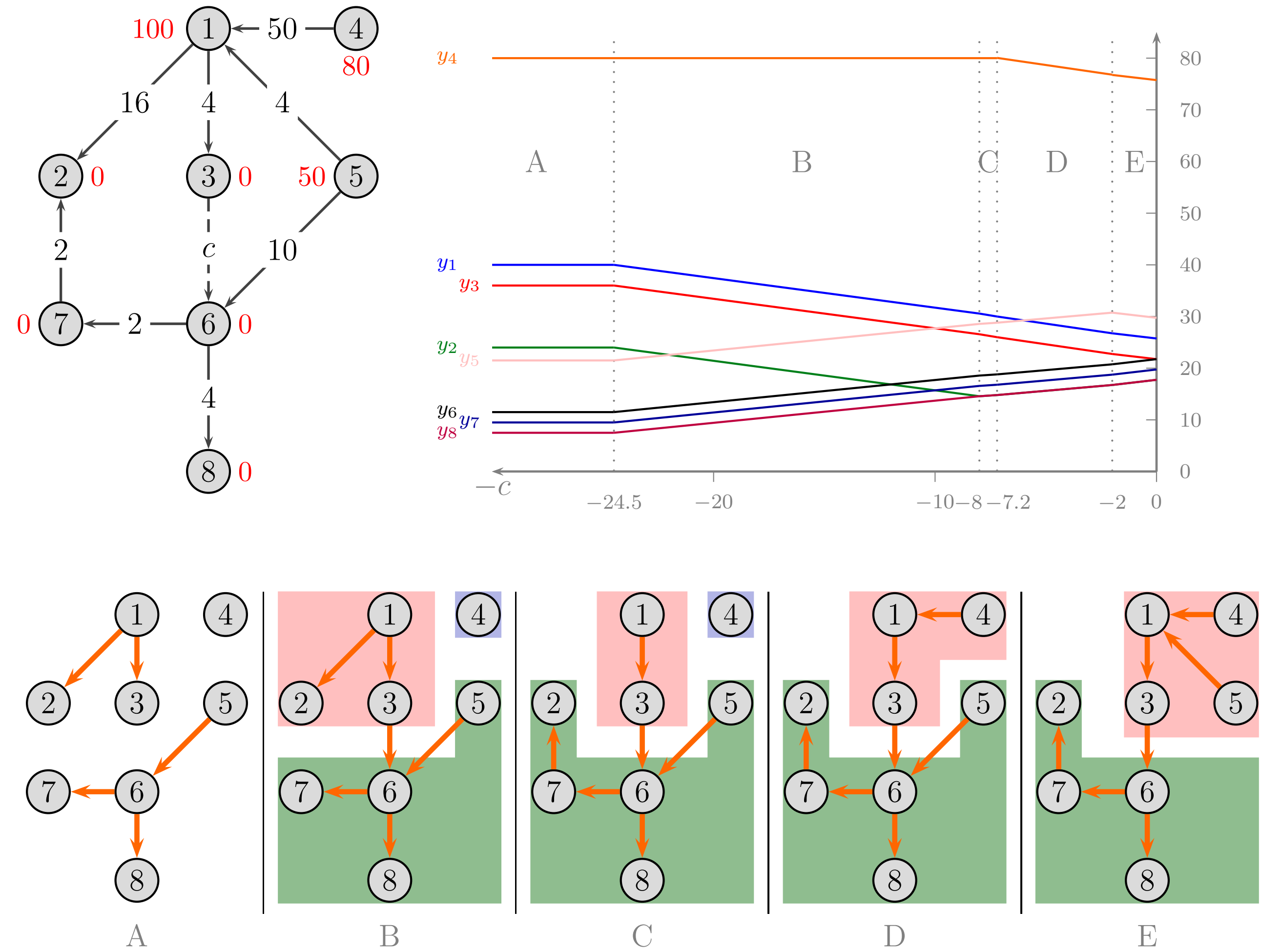

Figure 1: Changing altruism – an example. The top-left panel shows altruism levels as given by the "transfer cost" $-\ln \alpha_{ij}$, the comparative statics brings altruism level of agent 3 to agent 6 from 0 ($c = \infty$) to 1 ($c = 0$). The red figures are initial incomes. The top-right panel shows the evolution of consumption for all agents. The lower panels show the graphs of the transfer network in the different regions. In the green zones are agents connected to 6, whose consumption increases, in the red zones are agents connected to 3, whose consumption decreases, and in the blue zones are agents connected to neither, whose consumption is stable.

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