Dynamic Asset Pricing
Introduction to Microstructure

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Motivation

- **First part:** generalize the GE model to a setting with multiple periods.

- **Second Part:** look at some microstructure models, that model price formation, trading and insider information in more details.
Outline

• Setup: Uncertainty, Information and Securities

• Arbitrage, State Prices, Martingales and Risk-Neutral Measures

• Individual Optimality

• Equilibrium, Pareto Optimality and Asset Pricing

• Microstructure: Screening

• Microstructure: Signaling
Uncertainty

• $\Omega$ is a (finite) set of states.

• Uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$.

• $\mathcal{F}$ denotes the tribe of subsets of $\Omega$ that are events and can be assigned a probability.

• $P$ is a probability measure that assigns a probability $P(B)$ to any event $B \in \mathcal{F}$. 
Time and Information

- \( T + 1 \) dates: 0, 1, \cdots, \( T \).

- \( \mathcal{F}_t \subseteq \mathcal{F} \) is the set of events known to be true or false at time \( t \).

- Perfect memory assumption: \( t \leq t' \Rightarrow \mathcal{F}_t \subseteq \mathcal{F}_{t'} \).

- The filtration \( \mathbb{F} = \{ \mathcal{F}_0, \cdots, \mathcal{F}_T \} \) represents how information is revealed through time.

- An adapted process is a sequence \( X = \{ X_0, \cdots, X_T \} \) such that \( X_t \) is observable at \( t \). (set \( \mathcal{A} \))
  - Formally, \( X_t \) is a random variable with respect to \( (\Omega, \mathcal{F}_t) \)

- \( X \) is a martingale if for every \( t < s \):
  \[
  E( X_s | \mathcal{F}_t ) = X_t
  \]
  - Interpretation: the best prediction is that \( X \) will stay the same.
Example
Stopping Times

• A stopping time is a random variable $\tau$ taking values in $\{0, \cdots, T\} \cup \{\infty\}$ such that:

$$\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$$

• That is at time $t$ you know whether you have already stopped or not.

• Then $X$ is a martingale iff. for any stopping time $\tau$:

$$E(X_\tau) = E(X_0)$$

• We will see that martingales are useful in the characterization of security prices.
Securities and Dividends

- A security is a claim to an adapted dividend process $d$.
- Each security has an adapted security-price process $S$.
  - $S_t$ is the ex-dividend price of the security at $t$ (after payment of $d_t$).
Security Market

- The market consists in $K$ securities defined by the $K$-dimensional adapted process $d = (d^1, \ldots, d^K)$.

- A trading strategy is an adapted $K$-dimensional process $\theta$.

- $\theta_t = (\theta^1_t, \ldots, \theta^K_t)$ represents the portfolio held after trading at $t$.

- The dividend process $d^\theta$ generated by $\theta$ is defined by

$$d^\theta = \theta_{t-1} \cdot (S_t + d_t) - \theta_t \cdot S_t$$

- By convention $\theta_{-1} = 0$. 

Security Market

- $d_t$ is paid
- New portfolio $\theta_{t-1}$
- New prices $S_t$

- $d_{t+1}$ is paid
- New portfolio $\theta_t$
- New prices $S_{t+1}$
- New portfolio $\theta_{t+1}$
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Arbitrage

- Consider a dividend-price pair \((d, S)\) for \(K\) securities (a market).
- A trading strategy \(\theta\) is an arbitrage if \(d^\theta > 0\).
  - (This is the same notion as in Lecture 4)

**Proposition**

There is no arbitrage if and only if there exists an adapted process \(m \gg 0\) such that for any trading strategy \(\theta\)

\[
E \left( \sum_{t=0}^{T} m_t d_t^\theta \right) = 0.
\]

- The idea of the proof is the same as the existence of a positive state-price vector in the two-periods case.
State-Price Deflator

- An adapted process $m \gg 0$ is a state-price deflator (or pricing kernel) if for all $t$

$$S_t = \frac{1}{m_t} E_t \left( \sum_{s=t+1}^{T} m_s d_s \right)$$

- Note: then $m_{t+1}/m_t$ is the analog of the discount factor in the 2 periods model:

$$S_t = E_t \left( \frac{m_{t+1}}{m_t} \{ d_{t+1} + S_{t+1} \} \right)$$
State-Price Deflator

• It is easy to show that $m$ is a state-price deflator iff. for every $\theta$

$$\theta_t \cdot S_t = \frac{1}{m_t} E_t \left( \sum_{s=t+1}^{T} m_s d_s^{\theta} \right)$$

• At any time the market value of a trading strategy is the sum of the state-price discounted expected future dividends generated by the strategy.

• The gain process is defined by (your cumulative gains up to $t$):

$$G_t = S_t + \sum_{s=1}^{t} d_s$$

• Claim: $m \gg 0$ is a state price deflator iff. $S_T = 0$ and the deflated gain process defined by $G^m_t = m_t S_t + \sum_{s=1}^{t} m_s d_s$ is a martingale (see the problem set).
No Arbitrage

Theorem

\((d, S)\) admits no arbitrage iff. there is a state-price deflator \(m\).

Proof: \((\Rightarrow)\)

- There exists \(m \gg 0\) s.t. \(E \left( \sum_{t=0}^{T} m_t d_t^\theta \right) = 0\).
- Also \(S_T = 0\) (what is the arbitrage otherwise?)
- Consider a security \(k\) and a stopping time \(\tau \leq T\).
- Consider a strategy that trades only \(k\) such that \(\theta_t^k = \mathbb{1}_{t<\tau}\).
- Then:
  \[
  E \left( -S_0^k m_0 + \sum_{t=1}^{\tau} m_t d_t^k + m_{\tau} S_{\tau}^k \right) = 0
  \]
  \[
  \text{That is: } G_{0,m}^k = E \left( G_{\tau,m}^k \right) \Rightarrow G^m \text{ is a martingale.}
  \]

QED
Riskless Borrowing

• Now we introduce the bond.

• There is short-term riskless borrowing if for all $t < T - 1$ there is a trading strategy $\theta$ with $d_s^\theta = \mathbb{1}_{s=t+1}$.

• The associated short rate is the rate of return at $t$ of this strategy $1 + r_t = \frac{1}{\theta_t \cdot S_t}$.

• $\{r_t\}_{t=0}^T$ is the short-rate process.

• For the next results, we assume that there is short-term borrowing at a uniquely defined short-rate process $r$.

• For any $s \leq t$, define $R_{s,t} = (1 + r_s)(1 + r_{s+1}) \cdots (1 + r_{t-1})$.

• It is the return at time $t$ of a dollar invested at time $s$ and rolled over in the short term riskless strategy.
Equivalent Risk Neutral Measure

- With risk neutral investors only, the price of a security would be the discounted expected value of dividends. With risk averse investors, this cannot be true, but we can adjust the probability measure to get such a formula.

- $Q$ is an equivalent risk-neutral measure if for all $t < T$

$$S_t = E_t^Q \left( \sum_{s=t+1}^{T} \frac{d_s}{R_{t,s}} \right)$$

- Then for every trading strategy $\theta$ and all $t < T$

$$\theta_t \cdot S_t = E_t^Q \left( \sum_{s=t+1}^{T} \frac{d_s^\theta}{R_{t,s}} \right)$$

- Two probability measures $P$ and $Q$ are equivalent if they assign probability 0 to the same events.
No Arbitrage

**Theorem**

*There is no arbitrage iff. there exists an equivalent risk-neutral measure.*

**Proof:** (sketch)

- Consider the equivalent probability defined by

\[
\frac{dQ}{dP} = \frac{Q(\omega)}{P(\omega)} = \frac{m_T R_{0,T}}{m_0} \equiv \xi_T
\]

- Let \( \xi_t = E_t (\xi_T) \).

- Then for any process \( Z \) and \( s > t \),

\[
E_t^Q(Z_s) = \frac{1}{\xi_t} E_t (\xi_s Z_s)
\]
• Then consider the strategy \( \tilde{\theta} \) that consists in investing a dollar at time \( t \) and rolling it over in riskless borrowing until \( T \).

• That is \( \tilde{\theta}_t \cdot S_t = 1 \) and \( d^{\tilde{\theta}}_T = R_{T,t} \).

• The pricing relationship with a state-price deflator \( m \) gives an expression of \( \xi_t \):

\[
m_t = E_t (m_T R_{t,T}) = \frac{m_0 E_t \left( \frac{m_T R_{0,T}}{m_0} \right)}{R_{0,t}} = \frac{m_0 E_t \left( \xi_T \right)}{R_{0,t}} = \frac{\xi_t m_0}{R_{0,t}}
\]

• But then the pricing expression for any \( \theta \) is

\[
\theta_t \cdot S_t = \frac{1}{m_t} E_t \left( \sum_{s=t+1}^{T} m_s d^\theta_s \right) = E_t^Q \left( \sum_{s=t+1}^{T} \frac{d^\theta_s}{R_{t,s}} \right)
\]

QED
Complete Markets

• Markets are complete if for every adapted \( d \in \mathcal{A} \), there exists a trading strategy \( \theta \in \Theta \) such that \( d^\theta = d \).

**Proposition**

Suppose that \( \mathcal{F}_T = \mathcal{F} \) (everything is revealed in the final period) and there is no arbitrage. Then markets are complete if and only if there exists a unique equivalent risk-neutral measure.

Proof: (\( \Rightarrow \), \( \Leftarrow \) is more difficult)

• Suppose there are two equivalent risk-neutral measures \( Q_1, Q_2 \).

• Consider an event \( B \in \mathcal{F} \). By completeness, there exists a strategy \( \theta \) such that \( d_t^\theta = R_{0,T} \mathbb{1}_{t=T} \mathbb{1}_B \)

• Then: \( \theta_0 \cdot S_0 = Q_1(B) = Q_2(B) \).

QED
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Agents

- Introduce an individual agent with a strictly increasing VNM utility function $U$ over consumption.

- Consumption is described by a nonnegative adapted process $c = \{c_0, \cdots, c_T\}$. (set $A_+$)

- The agent is endowed with a nonnegative adapted endowment process $e = \{e_0, \cdots, e_T\}$.

- Then the budget-feasible consumption set of the agent is

  $$C(e) = \left\{ e + d^\theta \geq 0 \mid \theta \in \Theta \right\}$$

- Where $\Theta$ is the set of trading strategies.
Program of the Agent

• The utility of the agent over processes is

\[ U(c) = E \left( \sum_{t=0}^{T} \delta^t u(c_t) \right) \]

• The program of the agent is

\[ \sup_{c \in C(e)} E \left( \sum_{t=0}^{T} \delta^t u(c_t) \right) \]
Proposition

Suppose that the program of the investor has a solution $c^* \gg 0$. Then

(i) There is no arbitrage.

(ii) A state-price deflator is given by $m_t = \delta^t u'(c^*_t)$

(iii) For any $\tau \geq t + 1$

\[
S_t = \frac{1}{\delta^t u'(c^*_t)} E_t \left( \sum_{s=t+1}^{\tau} d_s \delta^s u'(c^*_s) + S_{\tau} \delta^\tau u'(c^*_\tau) \right)
\]

- This is as in the 2-periods case: you just need to look at the first-order condition in any direction $d^\theta$. 
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Equilibrium

- Agents $i = 1, \ldots, N$.

- An equilibrium is a collection $(\theta_1, \ldots, \theta_N, S)$ where:
  
  (i) $S$ is a security-price process.

  (ii) $\theta_i$ is an optimal trading strategy for agent $i$: it solves

  $$\sup_{\theta \in \Theta} U_i(c) \quad \text{s.t.} \quad c = e_i + d^\theta \in \mathcal{A}_+$$

  (iii) Markets clear:

  $$\sum_{i=1}^{N} \theta_i = 0.$$
Proposition

Suppose \((\theta_1, \cdots, \theta_N, S)\) is an equilibrium and markets are complete. Then the associated consumption allocation is Pareto optimal.

- Note that the completeness of markets depends on the price process \(S\) itself.
- This makes the existence of an equilibrium a nontrivial issue.
Representative Agent

• Consider the program \((P)\)

\[
u_\lambda(x_t) = \sup_{c_1, \ldots, c_n} \sum_{i=1}^{N} \lambda_i u_i(c_i) \quad \text{s.t.} \quad c_1 + \cdots + c_N \leq x_t
\]

**Proposition**

Suppose that \((\theta_1, \cdots, \theta_N, S)\) is an equilibrium and that markets are complete. Then there exists some \(\lambda > 0\) such that \((0, S)\) is a no-trade equilibrium for the one-agent economy \((U_\lambda, \bar{e}, d)\), where \(\bar{e} = \sum_i e_i\). Then the program \((P)\) at \(\lambda\) and \(x_t = \bar{e}_t\) is solved by the equilibrium consumption allocation at \(t\).
Asset Pricing

In this case, we can write for every $\tau \leq t$:

$$S_t = \frac{1}{\delta^t u'_\lambda(\bar{e}_t)} E_t \left( S_{\tau} \delta^\tau u'_\lambda(\bar{e}_\tau) + \sum_{s=t+1}^{\tau} d_s \delta^s u'_\lambda(\bar{e}_s) \right)$$
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Motivation

• In the GE model, all agents are price takers.

• This is not true in practice, on financial markets, prices are set by a market maker.

• Also, the details of trading can differ (Nasdaq, NYSE etc... ): what impact does it have?

• Questions: Are markets efficient? Does insider trading matter? How should the market maker set prices? Why do the bid and ask prices differ? What information is embedded in prices?
Microstructure Models

• Today we look at two workhorse models with sequential moves.

1. Screening models in which the market maker sets prices first (Glosten Milgrom, 1985): quote driven markets
   • The market maker must anticipate the meaning of buying/selling orders and protects herself against insider trading. Example: NASDAQ.

2. Signaling model in which the market maker sets prices ex post (Kyle, 1985): order driven markets
   • Then the orders placed by traders must be interpreted by the market maker. Example: market opening batch auction in Euronext.

• Many trading mechanisms are hybrids and mix features of both systems. Example: NYSE, London SE.
Glosten Milgrom 1985, Setup

- Dynamic trading model: the value of an asset $V \geq 0$ will become public at $T$.

- In each period $t = 0, \cdots, T$ the agents can trade the asset.

- Three types of agents, :
  - A continuum of informed traders with private information about $V$.
  - A continuum of uninformed traders.
  - Market makers or specialists, whose role is to organize trading, provide liquidity and set prices.
Order of Play

• Sequence of moves in a period:

1. A randomly selected trader arrives.

2. The specialists set prices: $A_t$ and $B_t$.
   • $A_t$ is the “ask” price at which the specialist is willing to sell the asset.
   • $B_t$ is the “bid” price at which the specialist is willing to buy the asset.

3. The trader places her orders: buy/sell one unit or do nothing.
Preferences

• All agents are risk neutral.

• Each agent has a utility $\rho \theta V + c$

• $\theta \in \{-1, 0, 1\}$ is the quantity of the asset purchased by the agent, $c$ is her current consumption.

• $\rho$ is an individual parameter capturing the trade off between current and future consumption. It is independent of $V$ and any information about $V$.

• For the specialist $\rho = 1$.

• Remark: $\rho$ must differ across agents or the no trade theorem prevents any trade.
Competition

• We assume that there is price competition between specialists.

• Hence specialists must set a price that gives them 0 profit.

• There exist models with monopolistic or oligopolistic market makers (but it’s harder to make them dynamic).
Information

• For an uninformed trader, upon arrival: any public information exogenously revealed so far, all past transaction prices. \( \rightarrow U_t \)

• For an informed trader: \( I_t \) finer than \( U_t \).

• For the specialist: \( S_t \) coarser than \( I_t \) and finer than \( U_t \).

• Before trading, the traders also observe prices \( A_t \) and \( B_t \).

• Let \( F_t \) represent the information of a trader so that:

\[
F_t = \begin{cases} 
U_t, A_t, B_t & \text{if uninformed} \\
I_t, A_t, B_t & \text{if informed}
\end{cases}
\]
Buy/Sell and Pricing Decision

• Let $Z_t = \rho_t E (V | \mathcal{F}_t)$ (So $Z_t$ differs for the informed and uninformed traders)

• The trader will buy if $Z_t > A_t$ and sell if $Z_t < B_t$. (Here we assume $A_t \geq B_t$ see the problem set for a proof that it is the case)

• The specialist’s expected profit from an arrival at time $t$ is therefore:

$$E \left( [A_t - V] \mathbb{1}_{Z_t > A_t} + [V - B_t] \mathbb{1}_{Z_t < B_t} | S_t \right)$$

$$= \left\{ A_t - E (V | S_t, Z_t > A_t) \right\} P (Z_t > A_t | S_t)$$

$$- \left\{ B_t - E (V | S_t, Z_t < B_t) \right\} P (Z_t < B_t | S_t)$$

• Then the zero profit condition implies:

$$A_t = E (V | S_t, Z_t > A_t) \quad \text{and} \quad B_t = E (V | S_t, Z_t < B_t)$$
**Equilibrium**

- Hence an equilibrium is characterized by:
  - **Zero Profit:**
    \[
    A_t = E(V \mid S_t, Z_t > A_t) \\
    B_t = E(V \mid S_t, Z_t < B_t)
    \]
  - \(A_t\) and \(B_t\) are measurable with respect to \(\mathcal{F}_t\) (the trader knows \(A_t\) and \(B_t\)).
  - The trader forms correct expectations: \(Z_t = \rho_t E(V \mid \mathcal{F}_t)\).
  - Existence is nontrivial in general, but here we will worry about some properties of equilibria.
Intuition

• Facing an informed seller is bad news for the specialist: she is probably making a loss on the trade.

• This must be factored into prices, as any trader may be informed. This is what conditioning on different events achieves.

• When a trader buys/sells it is good/bad news about the asset on average, so that should lead to a higher/lower expected value. The bid and ask price are these revised expected values.

• Because the informational content of a selling/buying order are different, the specialist sets different prices that incorporate the meaning of these events.

• Hence the bid/ask spread is a consequence of adverse selection.
Some Notations

- Let $\mathcal{K}_t = S_t \land U_t$ denote the set of events that are common knowledge at $t$, and let $E_t(.) = E(. \mid \mathcal{K}_t)$

- Then $A_t = E_t(A_t)$ and $B_t = E_t(B_t)$.

- And since $\rho_t$ is independent from $V$ and any information about $V$:

$$E_t(V \mid F_t, \rho_t) = E_t(V \mid F_t) \quad \text{and} \quad E_t(E_t(V \mid F_t) \mid \rho_t) = E_t(V)$$

- Also it is true that for any random variable $X$ and scalar $a$:

Fact 1- $E(X \mid X > a) \geq E(X)$, $>$ if $P(X > a) \in (0, 1)$

Fact 2- $E(X \mid X > a)$ non-decreasing in $a$, strictly increasing if on $\text{supp}(X)$. 
Proposition

Suppose there exists an equilibrium. Then:

\[ A_t \geq E_t(V) \geq B_t \]

And the inequalities are strict whenever adverse selection is possible, that is if:

\[ P\left(Z_t > E_t(V) \text{ and } E_t(V | \mathcal{F}_t) > E_t(V)\right) > 0 \]

and

\[ P\left(Z_t < E_t(V) \text{ and } E_t(V | \mathcal{F}_t) < E_t(V)\right) > 0 \]

• Note: adverse selection happens when the trader is willing to buy and the trader expects the asset to be worth more than what is common knowledge at time \( t \), or she is willing to sell and...
Proof:

• Let $C$ be the event:

\[ C = \{ Z > A \} = \{ E(V \mid \mathcal{F}) > A/\rho \} \]

• Then, by definition $A = E(V \mid S, C)$ so:

\[
A = E(A \mid C) = E( E(V \mid S, C) \mid C) = E(V \mid C) = E( E(V \mid C, \rho) \mid C ) \\
= E( E( E(V \mid \mathcal{F}, C, \rho) \mid C, \rho) \mid C ) = E( E( E(V \mid \mathcal{F}, \rho) \mid C, \rho) \mid C ) \\
= E( E( E(V \mid \mathcal{F}) \mid C, \rho) \mid C ) \geq E( E( E(V \mid \mathcal{F}) \mid \rho \mid C ) \\
= E( E(V \mid C) ) = E(V)
\]

• Where the inequality comes from Fact 1, and everything else is either from the law of iterated expectations or from independence.

\[
\text{QED}
\]
Efficient Markets Hypothesis

• What information is reflected in prices?

• Let $\mathcal{U}_t^+$ and $\mathcal{S}_t^+$ denote the information available just after the trade round at time $t$ is over.

• Sometimes there is no trade so let $T_k$, $k = 1, 2, \ldots$ be the times at which trade occurs and let $S_k = S_{T_k}^+$ and $\mathcal{U}_k = \mathcal{U}_{T_k}^+$.

• The price $p_k$ at which trade occurs at $T_k$ is:

$$p_k = \underbrace{A_{T_k}}_{E(V|S_{T_k}, Z_{T_k} > A_{T_k})} + \underbrace{B_{T_k}}_{E(V|S_{T_k}, Z_{T_k} < B_{T_k})} \mathbb{1}_{Z_{T_k} > A_{T_k}} + \mathbb{1}_{Z_{T_k} < B_{T_k}} = E(V|S_k)$$
Efficient Markets Hypothesis

Proposition

The sequence of transaction prices \( \{p_k\} \) forms a martingale relative to the specialist’s information \( \{S_k\} \) and the public information \( \{U_k\} \).

Proof:

\[
E(p_{k+1}|S_k) = E(E(V|S_{k+1})|S_k) = E(V|S_k) = p_k
\]

\[
E(p_{k+1}|U_k) = E(E(p_{k+1}|S_k)|U_k) = E(p_k|U_k) = p_k
\]

QED
Efficient Markets Hypothesis

- This means that the price after \( k \) transactions reflects all the public and specialist information until that transaction.

- This is slightly stronger than the semi-strong form of the efficient markets hypothesis since the information of the specialist is included.

- As soon as a trade is announced, the specialist and all the outsiders agree on the expected value of \( V \) since
  \[
  E(V|p_k) = E(V|E(V|S_k)) = E(V|S_k)
  \]
Bid/Ask Spread

• What are the determinants of the bid/ask spread?

• Here we assume: $U_t = S_t$ and insiders have no liquidity motive $\rho = 1$

**Proposition**

At any time $t$, $A_t$ increases and $B_t$ decreases whenever:

(i) The information of insiders at time $t$ becomes better (finer).

(ii) The ratio of informed to uninformed arrival rate at $t$ is increased.

(iii) The elasticity of uninformed supply and demand increases (roughly the uninformed trader is then more likely to trade).

• The general intuition is that the bid/ask spread becomes bigger whenever information asymmetries become stronger.
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Kyle 1985, Setup

- Asset liquidation value: $v \sim \mathcal{N}(p_0, \sigma_0^2)$

- Agents are risk neutral:
  - Insider who knows $v$ and submits orders of size $x$.
  - Noise traders who submit orders of total size $u \sim \mathcal{N}(0, \sigma_u^2)$ (independent of $v$).
  - Market maker sets price after observing the net order flow $y = x + u$.

- Timing:
  - **Stage 1**: informed and liquidity traders submit their orders.
  - **Stage 2**: the market maker sets the execution price.
Strategies and Equilibrium

• The informed trader places an order that depends on the liquidation value (known to her): \( x = \tilde{x}(v) \).

• The market maker sets her price as a function of the order flow: \( p = \tilde{p}(y) \).

• Knowing the strategy of the market maker, the goal of the trader is to maximize:

\[
\tilde{x}(v) = \arg \max_{\tilde{x}} x \left( v - E(\tilde{p}(x + u)) \right)
\]

• Knowing the strategy of the informed trader, the market maker sets her price to get 0 profit (competition):

\[
\tilde{p}(y) = E \left( v \mid \tilde{x}(v) + u = y \right)
\]

• Together, these two equations define an equilibrium.
Result

**Proposition**

There exists a unique equilibrium in linear strategies. In this equilibrium:

\[ \tilde{x}(v) = \frac{\sigma_u}{\sigma_0}(v - p_0), \]

and

\[ \tilde{p}(y) = p_0 + 2\frac{\sigma_0}{\sigma_u}y \]

- See the problem set for a proof.
Properties

- **Informativeness of prices:**
  - It can be measured by the quantity $\text{var}(v|p)$ where $p$ is the realized price.
  - We obtain that: $\text{var}(v|p) = \frac{1}{2}\sigma_0^2$ thus one half of the insider’s private information is incorporated into prices.
  - Also the volatility of prices is unaffected by the level of noise trading $\sigma_u^2$.

- The depth of the market is the change in order flow necessary to induce a price change of $1$:
  - It is equal to $\frac{\sigma_u}{2\sigma_0}$.
  - Hence the sensitivity of prices increases with the level of noise trading and the precision of the public information about prices.
Extensions

• Kyle (1985) has a dynamic version of this model, both with discrete and continuous time.

• Back and Baruch (2004) extend the Glosten and Milgrom (2004) model to consider a single informed trader who uses market orders and decides when to trade in continuous time. They show that this version of the model is close to the continuous time version of Kyle (1985) when uninformed traders arrive frequently and trade small quantities.
Insights

Some insights of this literature (from Vives, 2008):

• A large informed trader (insider) has incentives to trade slowly so as not to reveal too much information and keep her advantage.

• The insider will try to camouflage behind liquidity traders but has no incentive to introduce noise in his orders.

• Competition among insiders speeds up information revelation when they have symmetric information; otherwise informed traders may play a waiting game to try to induce their competitors to reveal information.

• Financial market anomalies such as momentum (short term positive autocorrelation) and reversal (long term negative autocorrelation) in stock returns can be explained with rational risk-averse traders.
Thanks!