Financial Economics
Equilibrium Asset Pricing

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Outline

• Setup, Intuition and Definitions
• Efficiency
• Optimality and Representative Agents
• The Efficient Markets Hypothesis
• Modeling Information
• Trade and Information
• Information and Prices
Setup: Consumers

• We consider an endowment economy:
  • No production.
  • At each period the whole aggregate is consumed or wasted.

• States of the world: \( s = 1, \cdots, S \).

• Several consumers \( i = 1, \cdots, N \) live two periods \( t \) and \( t + 1 \)
  • Subjective Probabilities: \( \pi^i \)
  • Utilities: \( U_i(c) = u_i(c_0) + \delta^i E_{\pi^i} u_i(c_1) \)
  • Endowment processes: \( e^i = (e^i_t, e^i_{t+1}(1), \cdots, e^i_{t+1}(S)) \gg 0 \)
Setup: Markets

- Markets are given by \((q, D)\) where \(D\) has \(K\) assets.

- Let \(C(q, D, e)\) be the set of (nonnegative) consumption processes attainable by a consumer with endowment process \(e\) in this market.

- To a consumption process of \(i\) is associated a portfolio \(\theta_i \in \mathbb{R}^K\).

- Let \(C^{AD}(p, e)\) be the set of (nonnegative) consumption processes attainable in an Arrow-Debreu economy with state-price vector \(p\).

- We know that: \(C(q, D, e) \subseteq C^{AD}(p, e)\)

- With complete markets: \(C(q, D, e) = C^{AD}(p, e)\)
Endowment Economy

• Is the endowment economy assumption problematic? No, as long as we don’t change the environment.
Trading Motives

• In an endowment (or pure exchange) economy, there are four nonexclusive reasons why people trade:

  • **Mutualization**: people do not have the same endowments and would like to smooth their consumption.

  • **Insurance**: people do not have the same levels of risk aversion, and the less risk-averse sell protection to the more risk-averse.

  • **Impatience**: people do not have the same levels of impatience, and the more patient lend to the more impatient.

  • **Speculation**: people have different beliefs (subjective probabilities) about the evolution of the stock market and wish to trade accordingly.

• In most models we will cut one of these sources of heterogeneity, and therefore annihilate some of these motives to trade in order to focus on a specific one.
Equilibrium Definition

An equilibrium is a price vector $q^*$ and a consumption process $c^*_i$ for each consumer such that:

- **Optimality:**
  \[ c^*_i \in \arg \max_{c \in C(q,D,e^i)} U_i(c) \quad (\forall i) \]

- **Market Clearing:**
  \[ \overline{c}^*_{t+1}(s) = \overline{e}_{t+1}(s) \quad (\forall s) \]

The equilibrium could also be formulated in terms of portfolio holdings $\theta^*_i$ for each agent, and then the market clearing condition would be:

\[ \overline{\theta}^*_{k} = 0 \quad (\forall k) \]

- $\overline{x} = \sum_{i=1}^{N} x^i$ denotes the aggregation operator.
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Efficiency

- An allocation $c$ defines a consumption process for every agent in the economy.
- It is feasible if $\bar{c}_t \leq \bar{e}_t$ and $\bar{c}_{t+1}(s) \leq \bar{e}_{t+1}(s)$ in every state $s$.
- We say that an allocation $c$ is (Pareto) efficient if it is feasible and there is no alternative feasible allocation $c'$ such that:
  \[ U_i(c') \geq U_i(c) \text{ for every } i, \text{ with at least one strict inequality.} \]
- Idea: no one can be made happier without hurting someone else.
- Let $C^{eff}$ denote the set of efficient allocations.
Complete Markets are Efficient

**Theorem**

If markets are complete and $c^*$ is an equilibrium allocation, then $c^* \in C^{\text{eff}}$.

- We know that complete markets are equivalent to an Arrow-Debreu economy, so this is just the first welfare theorem.
Proof:

- We can work with the Arrow-Debreu formulation.
- If $c^*$ is an equilibrium with corresponding prices $p_s$, then it is feasible by the market clearing condition.
- If $c$ Pareto dominates $c^*$, it must make every agent weakly better off.
- But then, by revealed preferences, it is either on the frontier or out of every agent's budget set:

\[ c_t^i + \sum_s p_s c_{t+1}^i(s) \geq e_t^i + \sum_s p_s e_{t+1}^i(s), \quad > \quad \text{for at least one } i \]

- But then

\[ \bar{c}_t + \sum_s p_s \bar{c}_{t+1}(s) > \bar{e}_t + \sum_s p_s \bar{e}_{t+1}(s) \]

- So $c$ cannot be feasible

QED
Social Planner

- How do complete financial markets compare to an omniscient utilitarian social planner?

\[ U_\lambda(x) \equiv \sup_c \sum_{i=1}^{N} \lambda_i U_i(c^i) \quad \text{subject to} \quad \bar{c} \leq x \]

**Proposition**

Suppose that \( U_i(.) \) is concave. Then a feasible allocation \( c^* \) is Pareto optimal iff. there exists \( \lambda > 0 \) such that

\[ c^* \in \arg \max_c \sum_{i=1}^{N} \lambda_i U_i(c^i) \quad \text{subject to} \quad \bar{c} \leq \bar{e} \]

- This is the program of a utilitarian social planner who puts weights \( \lambda \) on individuals in the population.
Comments

• Hence complete financial markets do as well as an omniscient planner.

• Of course we made some heroic assumptions about markets (completeness, rational consumers, no frictions or transaction costs).

• But then an omniscient social planner is a heroic assumption as well: in order to compute the optimal allocation she must know the preferences of all individuals in society.

• With markets, we only need individuals to know their own preferences.

• This is Hayek’s defense of markets.
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Representative Agents

• Aside from its allocational implications, Pareto optimality is convenient for asset pricing.

**Proposition**

Suppose that markets are complete and that \((\theta, q)\) is an equilibrium. Then there exists \(\lambda > 0\) such that \((0, q)\) is a no-trade equilibrium for the single agent economy \((U_\lambda(.), \bar{e}, D)\).

• Hence to price assets, we can use a representative agent economy with utility \(U_\lambda\).

• Question: What form for \(U_\lambda\)? How to interpret it?
Proof:

- Let \((c, p)\) be the corresponding Arrow-Debreu equilibrium.
- Let \(\psi = (1, p_1, \cdots, p_S)\).
- Let \(\alpha_i\) be \(i\)'s Lagrange multiplier. \(c^i\) maximizes

\[
U_i(x^i) - \alpha_i \left\{ \psi \cdot x^i - \psi \cdot e^i \right\},
\]

- But then for any feasible allocation \(x\), letting \(\lambda_i = 1/\alpha_i\)

\[
\sum_i \lambda_i U_i(c^i) \geq \sum_i \lambda_i \left[ U_i(x^i) - \alpha_i \left\{ \psi \cdot x^i - \psi \cdot e^i \right\} \right] \\
\geq \sum_i \lambda_i U_i(x^i) + \psi \cdot \sum_i (e^i - x^i) \\
\geq 0 \text{ by feasibility}
\]

\[
\sum_i \lambda_i U_i(x^i)
\]

- So \(c\) solves the optimal allocation problem for \(\lambda\).
• Then let us show that \((\bar{e}, p)\) is an equilibrium of the single-agent economy \((U_\lambda(\cdot), \bar{e})\).

• Suppose that \(\bar{e}\) does not solve the single-agent program.

• Then there exists \(\bar{x}\) such that \(U_\lambda(\bar{x}) > U_\lambda(\bar{e})\) and \(\psi \cdot \bar{x} \leq \psi \cdot \bar{e}\).

• But then there is an \((x^i)\) such that \(\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)\) and

\[
\sum_i \lambda_i \alpha_i \psi \cdot x_i \leq \sum_i \lambda_i \alpha_i \psi \cdot c^i
\]

• And then

\[
\sum_i \lambda_i \left\{ U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i) \right\} > \sum_i \lambda_i \left\{ U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i) \right\}
\]

• But that contradicts the optimality of \(c^i\) for at least one agent \(i\).

QED
Representative Agent Utility

• Even when agents are homogeneous there is no reason for $U_\lambda(.)$ to be the same as $U_i(.)$.

• This stronger property, called aggregation, holds when $U_i(.)$ is homothetic.

• So what can we say about $U_\lambda(.)$?

Lemma

If all agents have the same beliefs, the same impatience and $U_i(c) = u_i(c_t) + \delta E u_i(c_{t+1})$ with $u_i(.)$ continuously differentiable, then

$$U_\lambda(x) = u_\lambda(x_t) + \delta E u_\lambda(x_{t+1}),$$

where

$$u_\lambda(x) = \max_y \sum_i \lambda_i u_i(y^i) \quad \text{s.t.} \quad y^1 + \cdots + y^N \leq x,$$

and $u_\lambda(.)$ is continuously differentiable.
Equilibrium Asset Pricing

**Proposition**

*When markets are complete and agents are homogeneous in patience and beliefs, the state price vector is given by*

\[
p_s = \pi_s \frac{\delta u'_\lambda (\bar{e}_{t+1}(s))}{u'_\lambda (\bar{e}_t)}
\]

*And the price of any security \( d \) is given by*

\[
q(d) = E \left( \frac{\delta u'_\lambda (\bar{e}_{t+1})}{u'_\lambda (\bar{e}_t)} d \right)
\]
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Information

• We haven’t explicitly modeled information so far.

• If there is no private information then it is natural (is it?) to consider that the agents have the same objective probability distribution.

• Then equilibrium implies that

\[ q = E \left( \frac{\delta u'(\bar{e}_{t+1})}{u'(\bar{e}_t)} d \right) \]

• Where the expectation is taken with respect to the objective probability distribution.

• This means that all public information about the future is present in prices today.
Background

• It is difficult to predict the short term movements of prices.
• This is not surprising since if you could you would be able to make money out of it.
• However many investors try to do just that.
• Are they right to do it or is it impossible to beat the market?
The Question

• The heart of the question is whether we should make any judgement or always invest in an index fund.

• Or can stock experts make better investments than monkeys throwing darts at charts?

• Studies show that mutual funds do not consistently beat the market (average return across funds, and across time). Or at least do not do much better.
Efficient Markets Hypothesis

- **Weak Form**: Prices reflect all information contained in past price movements. If it is true, technical analysis is doomed.

- **Semi-Strong Form**: prices reflect all public information available.

- **Strong Form**: prices reflect all private information available. If it is true no one can gain from private information. Trading on inside information would be useless.

- These hypotheses seem extreme but they are part of the theoretical implications of the workhorse rational expectations model.

- On the other hand they do capture the fact that it is hard to make money on the stock market.
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Aumann’s Setup

• States of the world $\omega \in \Omega$.

• We can represent $i$’s information by a partition $P_i$ of $\Omega$.

• An event is a subset $E \subseteq \Omega$.

• $i$ knows $E$ at $\omega$, if $P_i(\omega) \subseteq E$.

• Example:
Meet and Join

- The join $P_1 \lor P_2$ is the coarsest common refinement of $P_1$ and $P_2$. It represents the pooled information of the agents.

- The meet $P_1 \land P_2$ is the finest common coarsening of $P_1$ and $P_2$. It represents what is common knowledge to both agents.
Common Knowledge

- Let $Q \equiv P_1 \lor \cdots \lor P_N$ and $R \equiv P_1 \land \cdots \land P_N$.

- An event $E$ is **common knowledge** if every one knows $E$, and everyone knows that everyone knows $E$, and so on ad infinitum.

**Definition (Aumann, 1976)**

An event $E$ is **common knowledge** at $\omega$ among agents $1, \cdots, N$ if $R(\omega) \subseteq E$. 
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Setup

- $S$ is the set of payoff-relevant states.
- $X$ is the set of payoff-irrelevant events: they do not affect endowments or tastes directly but may be statistically related to $s$.
- States of the world: $\Omega = S \times X$.
- Future endowments $e_i(s) \geq 0$.
- Utility over future consumption: $u_i(c, s)$ increasing and (weakly) concave in $c$ for every $s$.
- Subjective beliefs $\pi_i$ on $\Omega$, with full support ($\pi_i(\omega) > 0$ for every $\omega$).
- $E_i(.)$ is the expectation under $\pi_i$. 
Trades

- We assume that the agents can make trades contingent on $s$.
- That is: they can purchase portfolios of Arrow-Debreu securities for each $s$ (complete markets).
- Let $\theta_i$ be $i$’s portfolio. Then $\theta = (\theta_1, \cdots, \theta_N)$ is called an $s$-contingent trade.
  - A trade is feasible if markets clear, that is:
    \[
    \sum_i \theta_i = 0
    \]
  - It is mutually acceptable if for every $i$
    \[
    E_i(u_i(e_i + \theta_i)|i’s\ info) \geq E_i(u_i(e_i)|i’s\ info)
    \]
Concordant Beliefs

• Say that beliefs are concordant if:

\[ \pi_1(x|s) = \cdots = \pi_N(x|s) \]

• If you think of \( x \) as information about \( s \), then concordant beliefs mean that all agents interpret information in the same way: they agree on how information is generated for a given \( s \), but they may disagree on the likelihood of different \( s \).

• This is true in particular when agents have the same priors \( \pi_1 = \cdots = \pi_N \).
**ω- Trades**

- Trades could be contingent on \( x \) as well.
- We do not need to allow such trades but we will use them in the construction.
- Even if they were allowed, these trades would not be used.
- Indeed, suppose that \( \theta \) is an \( \omega \)-contingent feasible trade, then we can construct the \( s \)-contingent trade with portfolios

\[
\theta'_i(s) = E_i(\theta_i(\omega)|s).
\]

- Then \( \theta' \) is feasible if beliefs are concordant.
- And all investors prefer \( \theta'_i \) to \( \theta_i \):

\[
E_i(u_i(e_i + \theta'_i)) \geq E_i\left(E_i\left(u_i(e_i + \theta_i)|s\right)\right) = E_i(u_i(e_i + \theta_i))
\]

concavity
No-Trade Theorem

• Suppose that agents can trade after they are informed according to their partition.

• Assume that beliefs are concordant.

**Theorem (Milgrom and Stokey, 1982)**

If \( e \) is an ex ante Pareto optimal allocation, and it is common knowledge at \( \omega \) that \( \theta \) is a mutually acceptable feasible trade, then all investors are indifferent between \( \theta \) and the zero trade. If all investors are strictly risk averse, then \( \theta \) is the zero trade.

• In particular, if markets open a first time before the information is revealed, and then a second time after the information is revealed, agents do not trade the second time.
Intuition

“I’d never join a club that would have me for a member”

Groucho Marx

• The fact that the initial allocation is Pareto optimal means that agents have exhausted their insurance and consumption smoothing motives for trade.

• If they trade after receiving information, it is purely for speculative reasons: they want to take advantage of their information.

• Since they interpret information in the same way, they know that anyone who is willing to take their bet must have information that goes the other way.

• But then at least one of the parties should refuse the bet.
Proof:

• Suppose that it is common knowledge at $\omega' = (s', x')$ that $\theta$ is a feasible, mutually acceptable trade.

• Then, for every $\omega \in R(\omega')$ and every $i$:

$$E_i(u_i(s, e_i + \theta_i)|P_i(\omega)) \geq E_i(u_i(s, e_i)|P_i(\omega))$$

• Then consider the $\omega$-contingent portfolios $\theta^*_i = \theta_i \mathbb{1}_{R(\omega')}$. 

• $\theta^*$ is a feasible trade and from an ex ante perspective:

$$E_i(u_i(s, e_i + \theta^*_i)) = E_i\left( E_i\left(u_i(s, e_i + \theta_i \mathbb{1}_{R(\omega')})|P_i(\omega)\right)\right)$$

$$= E_i\left( E_i\left(u_i(s, e_i) \mathbb{1}_{\neg R(\omega')}|P_i(\omega)\right)\right)$$

$$+ E_i\left( E_i\left(u_i(s, e_i + \theta_i) \mathbb{1}_{R(\omega')}|P_i(\omega)\right)\right)$$
• But since $R$ is coarser than $P_i$, we have:

$$E_i(u_i(s, e_i + \theta_i^*)) = E_i\left( \mathbb{1}_{\neg R(\omega')} E_i\left( u_i(s, e_i|P_i(\omega)) \right) \right)$$

$$+ E_i\left( \mathbb{1}_{R(\omega')} E_i\left( u_i(s, e_i + \theta_i)|P_i(\omega) \right) \right)$$

$$\geq E_i\left( u_i(s, e_i)|P_i(\omega) \right) \text{ if } \omega \in R(\omega')$$

$$\geq E_i(u_i(s, e_i))$$

• Then $\theta_i^*$ is an $\omega$-contingent trade that was preferred to the zero trade by all investors ex ante.

• Then $\theta^{**} = (E_1(\theta_1^*|s), \cdots, E_N(\theta_N^*|s))$ is an $s$-contingent trade that is preferred by all ex ante. Because $e$ is Pareto optimal, all investors must be indifferent between $\theta^{**}$ and the zero trade.

QED
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Fully Revealing Equilibrium Prices

• Remember that $Q = P_1 \lor \cdots \lor P_N$ pools all the information of the agents.

Theorem (Milgrom and Stokey, 1982)

If $q(s)$ is a price vector that supports the Pareto optimal allocation $e$, and markets reopen after agents observe their private information, the following price vector at $\omega$:

$$q^*(s|x) = q(s)\pi(Q(\omega)|s)$$

together with the initial allocation constitutes a fully revealing rational expectations equilibrium.

• Note that $\pi(Q(\omega)|s)$ does not depend on $i$ because of concordant beliefs.

• Note also that the prices depend on $x$. 
Intuition

• This equilibrium price aggregates and reveals all the information available in the market.

• More precisely, it is the change in prices $\frac{q^*(s|x)}{q(s)} = \pi(Q(\omega)|s)$ that reveals the information.

• Hence in practice, all agents have the same information.

• This is the idea of strong efficiency: all private information is immediately revealed through prices.

• However, note that the fully revealing equilibrium is not necessarily unique.
Proof:

• Suppose the prices are given by \( \hat{q}(s|x) \).

• Then by observing the change in prices, each investor learns \( \pi(Q(\omega)|s) \).

• This is all they need for Bayesian updating conditional on all the available information:

\[
\pi_i(s|Q(\omega)) = \pi(Q(\omega)|s) \frac{\pi_i(s)}{\pi_i(Q(\omega))}
\]

• This gives the fully revealing nature of the equilibrium.

• It is easy to check that these prices together with \( e \) form an equilibrium (see the proof of the next result).

QED
Other Equilibria

• There can be other (less than fully revealing) equilibria, but in any equilibrium, information from an agent’s private signal is swamped by the information contained in prices.

**Theorem (Milgrom and Stokey, 1982)**

Assume that all agents are strictly risk averse and $e \gg 0$. By the first theorem $e$ is still an equilibrium allocation when the markets reopen. If $\hat{q}(s|x)$ is a supporting price for this equilibrium, then

$$\pi_i(s|P_i(\omega), \hat{q}) = \pi_i(s|\hat{q}).$$

• Hence, in any equilibrium, the information contained in prices trumps private information.
Proof:
• Since \((q, e)\) is an ex ante equilibrium:

\[
\frac{q(s)}{q(s')} = \frac{u_i'(s, e_i(s))}{u_i'(s', e_i(s'))} \frac{\pi_i(s)}{\pi_i(s')}
\]

• Since \((\hat{q}, e)\) is an ex post equilibrium

\[
\frac{\hat{q}(s|Q(\omega))}{\hat{q}(s'|Q(\omega))} = \frac{u_i'(s, e_i(s))}{u_i'(s', e_i(s'))} \frac{\pi_i(s|P_i(\omega), \hat{q})}{\pi_i(s'|P_i(\omega), \hat{q})}
\]

• Hence

\[
\frac{\pi_i(s|P_i(\omega), \hat{q})}{\pi_i(s'|P_i(\omega), \hat{q})} = \frac{\pi_i(s)}{\pi_i(s')} \frac{\frac{\hat{q}(s|Q(\omega))}{q(s)} \frac{q(s)}{q(s')}}{\hat{q}(s'|Q(\omega))}
\]

• Hence the ratios of the posteriors are independent from \(P_i(\omega)\).

• But since the probabilities sum to 1, the posteriors are independent from \(P_i\) as well.

QED
Comments

• Hence the theory of rational expectations leads to the prediction that markets satisfy strong efficiency.

• Yet even from a theoretical point of view, this conclusion is somewhat paradoxical.

• If investors cannot benefit from private information, why do they bother acquiring it? (Grossman, Stiglitz, 1980)

• If the only information aggregated by prices is “free”, there is no need for markets. (Hayek)
• Furthermore, if there is no trade how can there be prices?

• This last remark calls for more detailed models of price formation in financial markets (see the micro-structure literature).

• One solution is to assume that there is another source of uncertainty that makes prices noisy. This is the case if there are noise traders that trade assets for random motives and add noise to the supply and demand of financial assets.
Thanks!