Financial Economics
Choice under Uncertainty

Eduardo Perez-Richet

Ecole Polytechnique
Outline

• Choice under uncertainty.
• Expected Utility
• Representation Theorem
• Measuring Risk Aversion
• Comparing Risky Prospects
• Beyond Expected Utility: subjective probabilities and behavioral criticisms
• Application: the portfolio problem
Setup

• A set of possible prizes $\mathcal{X}$ (i.e. outcomes or consequences).
  • We assume that $|\mathcal{X}| = n < \infty$.

• A lottery is a probability distribution on $\mathcal{X}$: $p \in \Delta(\mathcal{X})$.

• $\Delta(\mathcal{X})$ is called the $n$-dimensional simplex. It is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n_+$:

$$\Delta(\mathcal{X}) = \left\{ p \in \mathbb{R}^n_+ : \sum_i p_i = 1 \right\}$$

• When we consider preferences under uncertainty, we are really talking about an order $\succeq$ on the set of lotteries $\Delta(\mathcal{X})$. 
Representing the Simplex

- With $n = 2$:

- With $n = 3$:
• $\Delta(\mathcal{X})$ is convex: for $\alpha \in [0, 1]$, $p'' = \alpha p + (1 - \alpha)p' \in \Delta(\mathcal{X})$.

• $p'' = \alpha p + (1 - \alpha)p'$ can be interpreted as a compound lottery: facing lottery $p$ with probability $\alpha$ and $p'$ with probability $1 - \alpha$. 
Preferences

• We consider preferences over lotteries rather than outcomes in order to account for uncertainty.

• Expected utility theory relies on preference orders $\succeq$ over $\Delta(\mathcal{X})$ that satisfy:
  
  - **Completeness:** $p \succeq p'$ or $p' \succeq p$
  - **Transitivity:** $p \succeq p'$ and $p' \succeq p'' \Rightarrow p \succeq p''$.
  - **Continuity**
  - **Independence**
Continuity Axiom

Definition (continuity)

\( \succeq \) is continuous if for every \( p_H \succeq p_M \succeq p_L \), there exists a scalar \( \alpha \in [0, 1] \) such that:

\[
\alpha p_H + (1 - \alpha) p_L \sim p_M
\]

Why continuity? By analogy with the intermediate value theorem. Consider the compounding function \( g(.) \). It satisfies a sort of intermediate value theorem with the order \( \succeq \).

\[
g : [0, 1] \rightarrow \Delta(\mathcal{X}) \\
\alpha \rightarrow \alpha p_H + (1 - \alpha) p_L
\]
Criticism and Intuition

• Example (Kreps): consider the following lotteries
  • $p_H :$1,000 for sure
  • $p_M :$10 for sure
  • $p_L :$ Get killed for sure

• The continuity axiom implies that there is a probability $\alpha$ of getting $1,000 and $1 - \alpha$ of dying that makes you indifferent between that bet and $10 for sure.

• On the other hand suppose you have the choice between:
  • Driving your car to your friend’s place and she will give you $1,000.
  • Staying home.

• Most people say they would drive, yet there is a positive probability of dying.
Independence Axiom

Definition (independence)
\[ \succeq \] satisfies independence if for every \( p_0 \) and \( \alpha \in [0, 1] \)

\[ p \succeq p' \]

\[ \iff \]

\[ \alpha p + (1 - \alpha)p_0 \succeq \alpha p' + (1 - \alpha)p_0 \]

- Compounding with a third independent lottery preserves the order between two lotteries.
Intuition

• The independence axiom seems quite sensible if you think about it in terms of compounding.

• Two possible states of the world tomorrow with given probabilities. In state 1 you always get the same, say beer, but in state 2 you have two possible prizes peanuts or pretzels.

• It is intuitive that when choosing, you should care only about your preferences between pretzels and peanuts.

• However compounding is only an interpretation of the meaning of the lottery \( p'' = \alpha p + (1 - \alpha)p' \).

• We will see that in experiments independence is often violated.
Outline

• Choice under uncertainty.

• Expected Utility

• Representation Theorem

• Measuring Risk Aversion

• Comparing Risky Prospects

• Beyond Expected Utility: subjective probabilities and behavioral criticisms

• Application: the portfolio problem
Expected Utility

**Definition (Expected Utility)**

A utility function $U : \Delta(\mathcal{X}) \to \mathbb{R}$ has an expected utility form (or is a von Neumann-Morgenstern utility function) if there exist numbers $(u_1, \cdots, u_n)$ for each of the prizes $(x_1, \cdots, x_n)$ such that

$$U(p) = \sum_{i=1}^{n} p_i u_i \quad (= p \cdot u = \mathbb{E}_p u)$$

- Hence if an agent has an expected utility function all we need to know to pin down her preferences over uncertain outcomes are her payoffs from the certain outcomes $x_1, \cdots, x_n$. 
Theorem

\[ U \text{ has an expected utility form } \iff \text{it is linear in probabilities} \]

\[ i.e. \ U(\alpha p + (1 - \alpha)p') = \alpha U(p) + (1 - \alpha)U(p') \]

Proof

⇒ is easy.

⇐ is a good exercise.
Indifference Curves

• Consider the case $n = 3$.

• Linearity implies that the indifference curves are **straight lines**.
  • Suppose $U(p) = U(p')$ and take $p''$ on the line between $p$ and $p'$
    \[
    U(p'') = U(\alpha p + (1 - \alpha)p') = \alpha U(p) + (1 - \alpha)U(p') = U(p)
    \]

• And they are also **parallel**
  • They must be orthogonal to the projection $\tilde{u}$ of $u$ on $\Delta(\mathcal{X})$
Indifference Curves

• More generally, linearity means that the indifference curves are \( n - 2 \) dimensional hyperplanes in the simplex.
  
  • The set of \( p \in \mathbb{R}^n \) that solve \( p \cdot u = c \) is an \( n - 1 \) dimensional hyperplane.

  • But we are considering it's intersection with the simplex which is a convex subset of an \( n - 1 \) dimensional hyperplane.

\[
p_1 p_2 \quad n=2
\]

\[
x_1 x_2 \quad n=3
\]
Outline

• Choice under uncertainty.

• Expected Utility

• Representation Theorem

• Measuring Risk Aversion

• Comparing Risky Prospects

• Beyond Expected Utility: subjective probabilities and behavioral criticisms

• Application: the portfolio problem
A Representation Theorem

• Reminder: $U(.)$ represents $\succeq$ iff. $a \succeq b \iff U(a) \geq U(b)$.

**Theorem (von Neumann-Morgenstern, 1947)**

Let $\succeq$ be complete and transitive. Then it satisfies **continuity and independence** iff. it admits an expected utility representation.

• To the extent that continuity and independence are acceptable axioms (not to mention completeness and transitivity), this is a remarkable result.

• $\iff$ is easy.

• $\Rightarrow$ the main idea is simple: given continuity, independence implies the same type of indifference curves as expected utility.
Intuition

• Getting the straight lines:
  
  • Suppose \( p \sim p' \) and let \( p'' \) be on the line between \( p \) and \( p' \).
  
  • Then, by independence

\[
p'' = \alpha p + (1 - \alpha)p' \sim \alpha p' + (1 - \alpha)p' = p'
\]

• Why they must be parallel:

Suppose \( p \sim p' \).

\( p'' \) on a non-parallel indiff. line

\[
p_0 \equiv p'' + \varepsilon(p'' - p)
\]

\[
\downarrow
\]

\[
p'' = \frac{1}{1+\varepsilon} p_0 + \frac{\varepsilon}{1+\varepsilon} p
\]

\sim

\[
p''' = \frac{1}{1+\varepsilon} p_0 + \frac{\varepsilon}{1+\varepsilon} p'
\]

Contradiction
Multiple Representations

Proposition

Suppose that $U$ is an expected utility representation of $\succeq$. Then $V$ is an expected utility representation of $\succeq$ iff. there exist scalars $a > 0$ and $b$ such that:

$$V(p) = aU(p) + b.$$

Remarks

• This means that we can always normalize the utility function by picking two outcomes $x_i$ and $X_j$ such that $\delta_{x_i} \succ \delta_{x_j}$ and fixing their values $u_i = 0$ and $u_j = 1$.

• For the proof $\iff$ is easy
Proof

⇒ For simplicity assume that the set of admissible lotteries is finite. Then there exists two lotteries \( \overline{p} \) and \( p \) such that for every admissible lottery \( p \)

\[
\overline{p} \succeq p \succeq \underline{p}
\]

Then for any \( p \) there exists a unique \( \lambda_p \in [0, 1] \) such that \( p \sim \lambda_p \overline{p} + (1 - \lambda_p) \underline{p} \). Then:

\[
U(p) = \lambda_p U(\overline{p}) + (1 - \lambda_p) U(\underline{p})
\]
\[
V(p) = \lambda_p V(\overline{p}) + (1 - \lambda_p) V(\underline{p})
\]

The first equation gives \( \lambda_p = \frac{U(p) - U(\underline{p})}{U(\overline{p}) - U(\underline{p})} \). And plugging back in the second equation, we get

\[
V(p) = \frac{V(\overline{p}) - V(p)}{U(\overline{p}) - U(\underline{p})} U(p) + V(p) - \frac{V(\overline{p}) - V(p)}{U(\overline{p}) - U(\underline{p})} U(p)
\]
• Choice under uncertainty.

• Expected Utility

• Representation Theorem

• Measuring Risk Aversion

• Comparing Risky Prospects

• Beyond Expected Utility: subjective probabilities and behavioral criticisms

• Application: the portfolio problem
Money Lotteries

- From now on we will work with lotteries on monetary prizes.
- Hence \( \mathcal{X} = \mathbb{R} \).
- \( \Delta(\mathcal{X}) \) is now the set of distributions on the real line.
- We will identify distributions by their cumulative density function (cdf) \( F(.) \).
  - \( F(x) \) is the probability that you receive a prize \( \leq x \).
  - \( F : \mathbb{R} \to [0, 1] \) is a cdf if it is non decreasing, and right continuous and \( \lim_{-\infty} F(x) = 0 \), and \( \lim_{\infty} F(x) = 1 \)
Expected Utility

- If $U$ is an expected utility, there exists a function $u(\cdot)$ such that:

$$U(F) = \int u(x) dF(x)$$

- $u : \mathbb{R} \rightarrow \mathbb{R}$ is sometimes called a Bernoulli utility function.

- We can also write $U(F) = E_F u(x)$
Risk Aversion

• Reminder: $\delta_y$ is the degenerate lottery that puts probability 1 on $y$.

• It is natural to define risk aversion as follows.

**Definition (Risk Aversion)**

A decision maker is (strictly) risk averse if for every non-degenerate lottery $F$ with expected value $E_F(x) = \int x dF(x)$, the decision maker (strictly) prefers the degenerate lottery $\delta_{E_F(x)}$ to $F$.

• Hence risk aversion says that for all $F$

$$u \left( \int x dF(x) \right) \geq \int u(x) dF(x)$$

• This mathematical expression is called Jensen’s inequality.
Risk Aversion

**Proposition**

A decision maker is *(strictly)* risk averse iff. $u(.)$ is *(strictly)* concave.

- If $u(.)$ is convex, Jensen’s inequality is reversed and we talk about **risk loving** preferences.

- If $u(.)$ is linear and the agent is **risk neutral**: she values a lottery as its expected monetary payoff.

- Of course, a utility function may exhibit none of these properties globally.
Graphic Intuition

$u(. \ldots) = u(E_F) = E_F u(x)$

$x_1 \quad E_F \quad x_2$

$\frac{1}{2}$ $1/2$

$F$
Certain Equivalent

- For a risk-averse agent $E_F(x)$ is preferred to $F$. But how many certain dollars is $F$ worth?

**Definition**

The certain equivalent $c(F, u)$ is the amount of dollars such that

$$u(c(F, u)) = \int u(x) dF(x)$$

- With risk aversion $c(F, u) \leq E_F(x)$.
- The difference $E_F(x) - c(F, u)$ can be thought of as a risk premium.
- It is a possible measure of the degree of risk aversion.
Local Risk Premium: an Approximation

- Consider a lottery $F$ over a small support. Then a realized prize $x$ is always close to $E_F$. We can write a Taylor approximation:

$$ u(x) = u(E_F) + u'(E_F)(x - E_F) + \frac{1}{2} u''(E_F)(x - E_F)^2 $$

\[ \Downarrow \]

$$ E_F u(x) = u(E_F) + \frac{1}{2} u''(E_F) \text{Var}_F $$

- We can also write the approximation

$$ u(c(F, u)) = u(E_F) + u'(E_F)(c(F, u) - E_F) $$

- Hence, combining these two:

$$ \text{RP}(F, u) = E_F - c(F, u) = -\frac{1}{2} \frac{u''(E_F)}{u'(E_F)} \text{Var}_F $$

- This is an approximation. The variance is not enough to characterize the riskiness of a lottery. Right concept = SOSD.
Absolute Risk Aversion

**Definition (Arrow-Pratt coefficient)**

For any twice differentiable $u(.)$, the **Arrow-Pratt coefficient of absolute risk aversion** is defined by

$$A(x, u) = -\frac{u''(x)}{u'(x)}$$

- $A(x)$ isolates the part of the local risk premium that comes from the preferences.
- The next natural step is to compare risk aversion globally: when are preferences represented by $u$ more risk averse than preferences represented by $v$. 
Comparing Risk Aversion

- It seems quite natural to say that $u$ is more risk averse than $v$ if for every $F$, $c(F, u) \leq c(F, v)$. What are the other ways to think about this comparison

**Proposition (Pratt, 1964)**

The following definitions of $u$ being *more risk averse* than $v$ are equivalent

1. *Whenever $u$ prefers a lottery $F$ to a certain payoff $\delta_x$, then so does $v$.*

2. *For every $F$, $c(F, u) \leq c(F, v)$*

3. *$u$ is more concave than $v$ in the sense that there exists an increasing concave function $g$ such that $u = g \circ v$.*

4. *For every $x$, $A(x, u) \geq A(x, v)$*
Proof:

1 ⇔ 2 should be clear.

2 ⇔ 3

\[ \begin{align*}
    c(F, u) & \leq c(F, v) \\
    \iff u(c(F, u)) & \leq u(c(F, v)) \\
    \iff \int g(v(x)) dF(x) & \leq g(\int v(x) dF(x))
\end{align*} \]

which is Jensen’s inequality for \( g \).

3 ⇔ 4 Taking derivatives

\[ \frac{u''}{u'} = \frac{v''}{v'} + v' \frac{g'' \circ v}{g' \circ v} \leq \frac{v''}{v} \]

QED
Risk Preferences and Wealth

- Suppose you and Bill Gates are identical, except for the additional $50 billions in his savings account.
- Suppose you prefer a given gamble to $100 for sure.
- It seems reasonable to assume that Bill would also prefer that gamble.
- This is the idea of Decreasing Absolute Risk Aversion (DARA).
- An individual has decreasing (constant, increasing) absolute risk aversion if $A(x)$ is decreasing (constant, increasing).
Relative Risk Aversion

• We can also look at the attitude towards multiplicative risk: get a stochastic return $z$ on your wealth

**Definition (Coefficient of Relative Risk Aversion)**

The coefficient of relative risk aversion is given by

$$R(x, u) \equiv -x \frac{u''(x)}{u'(x)} = xA(x, u)$$
Outline

• Choice under uncertainty.

• Expected Utility

• Representation Theorem

• Measuring Risk Aversion

• Comparing Risky Prospects

• Beyond Expected Utility: subjective probabilities and behavioral criticisms

• Application: the portfolio problem
Comparing Lotteries

- So far we’ve compared preferences.
- It is natural to assume \( u(\cdot) \) nondecreasing.
- Risk aversion (concavity) is often a reasonable requirement as well.
- What do these restrictions imply about the way lotteries are ranked?
- Can we compare the riskiness of two lotteries?
First Order Stochastic Dominance

**Definition (FOSD)**

$G(.)$ first order stochastically dominates $F(.)$ if for every nondecreasing Bernoulli utility function $u(.)$

\[
\int u(x) \, dG(x) \geq \int u(x) \, dF(x)
\]

• Equivalently: $E_G u(x) \geq E_F u(x)$
First Order Stochastic Dominance

Proposition

$G(.)$ first order stochastically dominates $F(.)$ iff. for every $x$

$$G(x) \leq F(x)$$

- The probability of getting less than $x$ is always lower under $G$.
- FOSD defines an incomplete order over the set of lotteries.
Proof:

• For simplicity, assume that \( u(.) \) is differentiable and that \( F \) and \( G \) have the same support \([x, \bar{x}]\).

• Then integrate by part:

\[
\int u(x) dG(x) = u(\bar{x}) - \int u'(x) G(x) \, dx
\]

• Hence

\[
\int u(x) dG(x) - \int u(x) dF(x) = \int u'(x) (F(x) - G(x)) \, dx
\]

• Remember \( u' \geq 0 \)

• Suppose \( F(x) < G(x) \) on some open ball \( B \), then we can pick \( u' > 0 \) on \( B \) and \( u' = 0 \) elsewhere.

QED
Intuition

- Suppose we start with a lottery $F$, and if a certain realization $x$ is reached we hold a second lottery that can potentially increase but never decrease $x$. Then the second lottery FOSD $F$.
- It’s the idea that FOSD shifts weight towards higher payoffs.
- Example:
Second Order Stochastic Dominance

**Definition (SOSD)**

Consider two distribution $F$ and $G$ with the same mean. $G$ second order stochastically dominates $F$ if for every concave Bernoulli utility function $u(.)$

\[
\int u(x) dG(x) \geq \int u(x) dF(x)
\]

- $G$ is preferred to $F$ by any risk averse individual.
Second Order Stochastic Dominance

Proposition

\( G(\cdot) \) second order stochastically dominates \( F(\cdot) \) iff. for every \( x \)

\[
\int_{-\infty}^{x} G(y) \, dy \leq \int_{-\infty}^{x} F(y) \, dy
\]

- The SOSD order is the natural way of ordering lotteries by their riskiness.
- It is an incomplete order.
Proof:

• Go back to the proof of FOSD and integrate by part a second time.

\[
\int u(x) dG(x) - \int u(x) dF(x) = u''(1) \int \{ F(x) - G(x) \} dx \\
+ \int u''(x) \int_{-\infty}^{x} \{ G(y) - F(y) \} dy dx
\]

• Integrating by part, and remembering that $F$ and $G$ have the same mean:

\[
\int \{ F(x) - G(x) \} dx = \int x dG(x) - \int x dF(x) = 0
\]

• Remember $u'' \leq 0$

• Suppose $\int_{-\infty}^{x} F(y) dy < \int_{-\infty}^{x} F(y) dy$ on some open ball $B$, then we can pick $u'' < 0$ on $B$ and $u'' = 0$ elsewhere.

QED
Mean Preserving Spread

- $G$ is a **mean preserving spread** of $F$ if it is obtained by compounding $F$ with lotteries of mean 0: conditional on reaching an $x$, you hold a new lottery of mean 0 and add the realization $y$ to $x$. (Rotschild-Stiglitz, 1970)

- You can show that $F \succeq_{\text{SOSD}} G$ iff. $G$ can be obtained through a series of mean preserving spreads of $F$.

- **Example:**

![Diagram](attachment:image.png)
Outline

- Choice under uncertainty.
- Expected Utility
- Representation Theorem
- Measuring Risk Aversion
- Comparing Risky Prospects
- Beyond Expected Utility: subjective probabilities and behavioral criticisms
- Application: the portfolio problem
Subjective Probabilities

• The theory of expected utility assumes the existence of well-defined objective risks: we considered preferences over a well defined object, lotteries.

• In reality, two people facing an uncertain situation may disagree on which “lottery” they are facing: for example win $1000 if France wins the rugby world cup.

• Intuitively, we should try to order these types of “bets”.
Subjective Probabilities

- Let $S$ be a set of states of the world (for example, France winning the cup).
- Let $\mathcal{X}$ be a set of prizes or outcomes (a cash flow for example).
- The bets that you face when you go to the bookmaker are mappings from states to prizes $f: S \rightarrow \mathcal{X}$.
- We call them acts.
- Then we can think of preferences over acts: which bet would you choose?
- Intuitively, these preferences involve a mix of beliefs and preferences over outcomes.
Subjective Preferences

- Savage (1954) shows that, under a set of (reasonable) axioms, I must choose as an expected utility maximizer who has in mind
  - A probability distribution over states of the world \( p \in \Delta(S) \) (my belief)
  - A utility function over prizes \( u: \mathcal{X} \to \mathbb{R} \).

- That is
  \[
  f \succeq g \iff \sum_{s \in S} p(s)u(f(s)) \geq \sum_{s \in S} p(s)u(g(s))
  \]

- This is an important result. In practice it means that we can work with subjective probabilities in the vNM utility functions.
An Experiment (Allais, 1953)

Choice 1

<table>
<thead>
<tr>
<th></th>
<th>$0</th>
<th>$48,000</th>
<th>$50,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lottery A</td>
<td>1%</td>
<td>66%</td>
<td>33%</td>
</tr>
<tr>
<td>Lottery B</td>
<td></td>
<td>100%</td>
<td></td>
</tr>
</tbody>
</table>

Choice 2

<table>
<thead>
<tr>
<th></th>
<th>$0</th>
<th>$48,000</th>
<th>$50,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lottery C</td>
<td>67%</td>
<td></td>
<td>33%</td>
</tr>
<tr>
<td>Lottery D</td>
<td>66%</td>
<td>34%</td>
<td></td>
</tr>
</tbody>
</table>
Problem with Independence

• Most people choose 1.B and 2.C.

• This is a violation of the independence axiom.

• Normalizing $u(0) = 0$, choice 1 implies:

$$0.34u(48) > 0.33u(50)$$

• Choice 2 implies:

$$0.34u(48) < 0.33u(50)$$

• Kahneman and Tverski’s explanation: people tend to overvalue certain outcomes (certainty effect).
Problems with Risk Aversion

• Would you accept a fair 50/50 bet where you win $1,050 or lose $1,000?

• If you would reject this bet regardless of your wealth, the degree of curvature of your utility function implies that you would also reject a 50/50 bet where you lose $20,000 or win any amount of money (Rabin, 2000)

• People seem to be excessively averse to small stakes gambles.

• The theory of loss aversion (Kahneman-Tversky, 1979) can account for this type of behavior.
Risk and Uncertainty

• An old idea is that a distinction should be drawn between risk (situations in which probabilities can be assigned to outcomes) and uncertainty (situations where one is just clueless).

• According to this definition the vNM theory deals with risk.

• With Savage, there is the idea that people always behave as if they could assign probabilities.

• Ellsberg’s experiment (1961) suggests that all uncertainty cannot be reduced to risk.

• Hence a different model may be needed to deal with this type of uncertainty (see the literature on ambiguity aversion).
Ellsberg’s Experiment

- **Urn A**: 49 black balls, 51 white balls.
- **Urn B**: 100 balls, black or white.
- You can choose between:
  1. Win $100 if you get a black ball from urn A.
  2. Win $100 if you get a black ball from urn B.
  3. Win $100 if you get a white ball from urn B
- People tend to choose (1), but this is incompatible with the assignment of probabilities to events.
Outline

- Choice under uncertainty.
- Expected Utility
- Representation Theorem
- Measuring Risk Aversion
- Comparing Risky Prospects
- Beyond Expected Utility: subjective probabilities and behavioral criticisms
- Application: the portfolio problem
A risk averse investor with wealth $w$ can decide how to invest it among two assets:

- A riskless asset with return $r$.
- A risky asset with return $z$, following the distribution $F$.

The agent only cares about her future wealth.

Her Bernoulli utility function $u(\cdot)$ is smooth.

Let $0 \leq \theta \leq w$ be the amount she invests in the risky asset.
Investor Program

• The program of the investor is

$$\max_{0 \leq \theta \leq w} E u(wr + \theta(z - r)) = \int u(wr + \theta(z - r)) \, dF(z)$$

• The marginal benefit of increasing $\theta$ locally is given by

$$\phi(\theta) \equiv \int (z - r)u'(wr + \theta(z - r)) \, dF(z)$$

• By concavity of $u$, $\phi(.)$ is decreasing.
Proposition

The investor invests a positive amount in the risky asset iff. it has a higher expected return than the riskless asset

\[ \theta > 0 \iff E(z) > r \]

Proof:

\[ \phi(0) = \int (z - r)u'(wr)dF(z) = u'(wr)(Ez - r) \]

QED

- This is true regardless of the degree of risk aversion: as long as the risky asset has a high expected return it is optimal to invest something in it.
Proposition

Consider two investors with preferences $u$ and $v$ such that $u$ is more risk averse than $v$. Then $v$ will invest more in the risky asset.

Proof:

- There exists a nondecreasing and convex function $h$ such that $v = h \circ u$.

- We can define $\phi_u(\theta)$ and $\phi_v(\theta)$. They are both decreasing, and the optimal level is at their crossing with 0, or no investment if they are always below 0.

- We will show that $\phi_u(\theta) \geq 0 \implies \phi_v(\theta) \geq 0$, and that will prove the result.
- We have:

\[
\phi_V(\theta) = \int (z - r)u'(wr + \theta(z - r)) h' \circ u(wr + \theta(z - r)) \mathrm{d}F(z)
\]

- Separate the right-hand side term into two terms:

\[
\int_{z<r} (z - r)u'(wr + \theta(z - r)) h' \circ u(wr + \theta(z - r)) \mathrm{d}F(z)
\]

\[
\uparrow z \leq 0
\]

\[
\geq h' \circ u(wr) \int_{z<r} (z - r)u'(wr + \theta(z - r)) \mathrm{d}F(z)
\]

and

\[
\int_{z\geq r} (z - r)u'(wr + \theta(z - r)) h' \circ u(wr + \theta(z - r)) \mathrm{d}F(z)
\]

\[
\uparrow z \geq 0
\]

\[
\geq h' \circ u(wr) \int_{z\geq r} (z - r)u'(wr + \theta(z - r)) \mathrm{d}F(z)
\]
• Reassembling terms, we have:

\[ \phi_v(\theta) \geq h' \circ u(wr) \int (z-r)u'(wr+\theta(z-r)) \, dF(z) = h' \circ u(wr) \phi_u(\theta). \]

• Hence \( \phi_u(\theta) \geq 0 \Rightarrow \phi_v(\theta) \geq 0 \) as we wanted.

• Therefore when it is beneficial to increase \( \theta \) under \( u \), it is also beneficial to increase it under \( v \), hence \( \theta_v \geq \theta_u \).

QED
Comparative Statics: Assets

• Hence we know how optimal investment changes when preferences change.

• The next natural step would be to study how optimal investment changes with the distribution of returns.

• But this turns out to be a bit more complicated (see problem set).

• Also it would be interesting to know what happens with multiple assets, but this is complicated too (see problem set).
Thanks!